1969

Influence of foundation characteristics on response of reactor containment structures to ground motions

Franz Peter Schauer
Iowa State University

70-7734

SCHAUER, Franz Peter, 1932-
INFLUENCE OF FOUNDATION CHARACTERISTICS
ON RESPONSE OF REACTOR CONTAINMENT
STRUCTURES TO GROUND MOTIONS.

Iowa State University, Ph.D., 1969
Engineering, civil

University Microfilms, Inc., Ann Arbor, Michigan

INFLUENCE OF FOUNDATION CHARACTERISTICS ON RESPONSE OF REACTOR CONTAINMENT STRUCTURES TO GROUND MOTIONS

by

Franz Peter Schauer

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Structural Engineering

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State University
Of Science and Technology
Ames, Iowa

1969

TABLE OF CONTENTS

I. INTRODUCTION

It is the stated object of this dissertation to investigate the influence of variation of foundation stiffness and foundation damping on the performance of nuclear containment type structures undergoing ground motions such as seismic disturbances. Foundation stiffness and foundation damping are thought to be the significant foundation variables with respect to the influence of the foundation on structural response.

Foundation soil types can range widely from conditions corresponding to bed rock to conditions corresponding to foundations of very soft clay or silt. The stiffness and damping characteristics associated with these foundation conditions are correspondingly wide ranging. For example, foundation stiffness may vary from one Kg/cm^3 to 100 Kg/cm^3 and higher. The influence of such wide ranging conditions on the response of structures to dynamic forces is not well understood. With particular reference to structural design against seismic ground motions, lack of definitive information has been a serious impediment to accurate structural design analysis.

Current practice in the design analysis of buildings does not take into account the underlying foundation and its properties. The current edition of the Uniform Building Code, for example, incorporates no provisions for inclusion of underlying foundation characteristics when considering the dynamic behavior of buildings. As a consequence current design practice for ground based structures is generally one of disregarding the foundation situation and considering, for analytical purposes, that the structure being designed is rigidly attached to its foundation.

While the aforementioned approach is, at first glance, unjustifiably crude it has the obvious merit of lending considerable simplification to the analysis and, as will be noted in reading the current literature, is not entirely without basis for the typical multistoried building frame of modern curtain wall construction supported on spread type footings (26). When the particular structure to be analyzed is stiff, not flexible, and is constructed integrally with a raft or floating foundation, however, a potential exists for considerable change in structural response due to foundation action. A typical example of this type of structure would be the multistoried shear wall building erected on a raft type foundation because of poor soil conditions.

Another type of structure of considerable stiffness and supported on a raft or slab foundation is the nuclear reactor containment structure. Typical of present day types of containments are those shown in figures one through five. Figure one represents in outline a prestressed concrete containment for a rather small (perhaps 400 MWe) pressurized water reactor system. Figure two represents conceptually the same type of containment for a larger pressurized reactor system (perhaps 900 MWe). Figures three and four are the reinforced concrete counterparts of systems represented by figures one and two, respectively. Figure five represents in outline a typical boiling water reactor unit of medium size (perhaps 600 MWe). The type I, II, III, and IV structures sketched in figures one through four are shell structures of simple geometry. They are fairly accurately modeled by just a few lump masses. The type V structure sketched in figure five is a massive rectangular frame structure of complexed geometry. It requires a considerable number of lumped masses for an accurate dynamic analysis.

The potentially extremely serious consequences of failure of nuclear containment structures while undergoing seismic loading has required that a detailed dynamic analysis be made of each such structure to verify its capability to withstand seismic design loads. In such analyses a typical procedure employed by the designer proceeds in the following fashion. The structure is first idealized as a lumped mass system. Generalized stiffnesses are assigned between masses and between the foundation mass and the supporting subsoil to include rocking and translational stiffnesses (typical containment structures are shown in figures one through five and their conventional idealization is shown in figure six). Next, the non-damped eigenvalues and eigenvectors are determined including, in general, a rotational coordinate. To each mode of vibration a damping percentage is assigned and each mode shape is treated in the subsequent steps as an equivalent single degree-of-freedom system. Using a response spectrum procedure or an actual earthquake time history the modal response for each mode is evaluated. Finally, the total response of the structure as a function of time or as a selected combination of modal maxima is secured.

As seen from the above description an attempt is presently being made to incorporate foundation conditions into the determination of nuclear structure response. However, three problems of fundamental nature impede the analysis and continue to cast doubt onto the validity of the results. First, the damping percentages to assign to individual vibrational modes are not known. Even for standard structural types the designer must rely on damping values obtained from forced vibration of only a limited number of structures at very low force levels that do not develop realistic damping action since, as is commonly recognized, more damping is developed as cracking and other damping mechanisms are brought into play at higher force levels. Second, the stiffness and damping coefficients for foundation translation and rocking are not known with any great degree of certainty because of the little research effort that has, to the present, been devoted to obtaining such coefficients. Lastly, the modal analysis method currently extensively used has as its theoretical basis the decoupling afforded the equations of motion of an undamped system through the use of a linear transformation.

Where a damped structural system is to be analyzed this decoupling is obtainable only when the damping matrix coefficients are term by term proportional to the coefficients of either the mass matrix or the stiffness matrix or a linear combination of these two matrices. Such damping is then termed proportional. However, for structural systems the requirement of proportional damping is a very restrictive one. Significantly large, pronouncely nonproportional damping is developed in structural systems where structure-foundation interaction is present.

As seen from the proceeding brief account of current analysis procedures and areas of uncertainity that exist in application of these procedures to problems where the foundation is potentially involved in the action of the structure, considerable research investigation is required in a number of areas before dynamic analysis of the complete structure-foundation system is on a sound analytical basis. It is the purpose of this dissertation to contribute to putting this area of analysis on a firmer basis by investigating

a) the role that the foundation conditions of stiffness and damping can play in moderating (or amplifying) the stresses in the superstructure and

b) the validity of current procedures in computing superstructure stress for varying magnitudes of foundation stiffness and damping.

The scope of this investigation is further restricted to include only reactor containment structures. Such structures are large and stiff and possess raft type foundations. They receive as a matter of routine detailed dynamic analyses of the type previously described. Their design is of considerable current interest due to the rapid expansion of the nuclear power program.

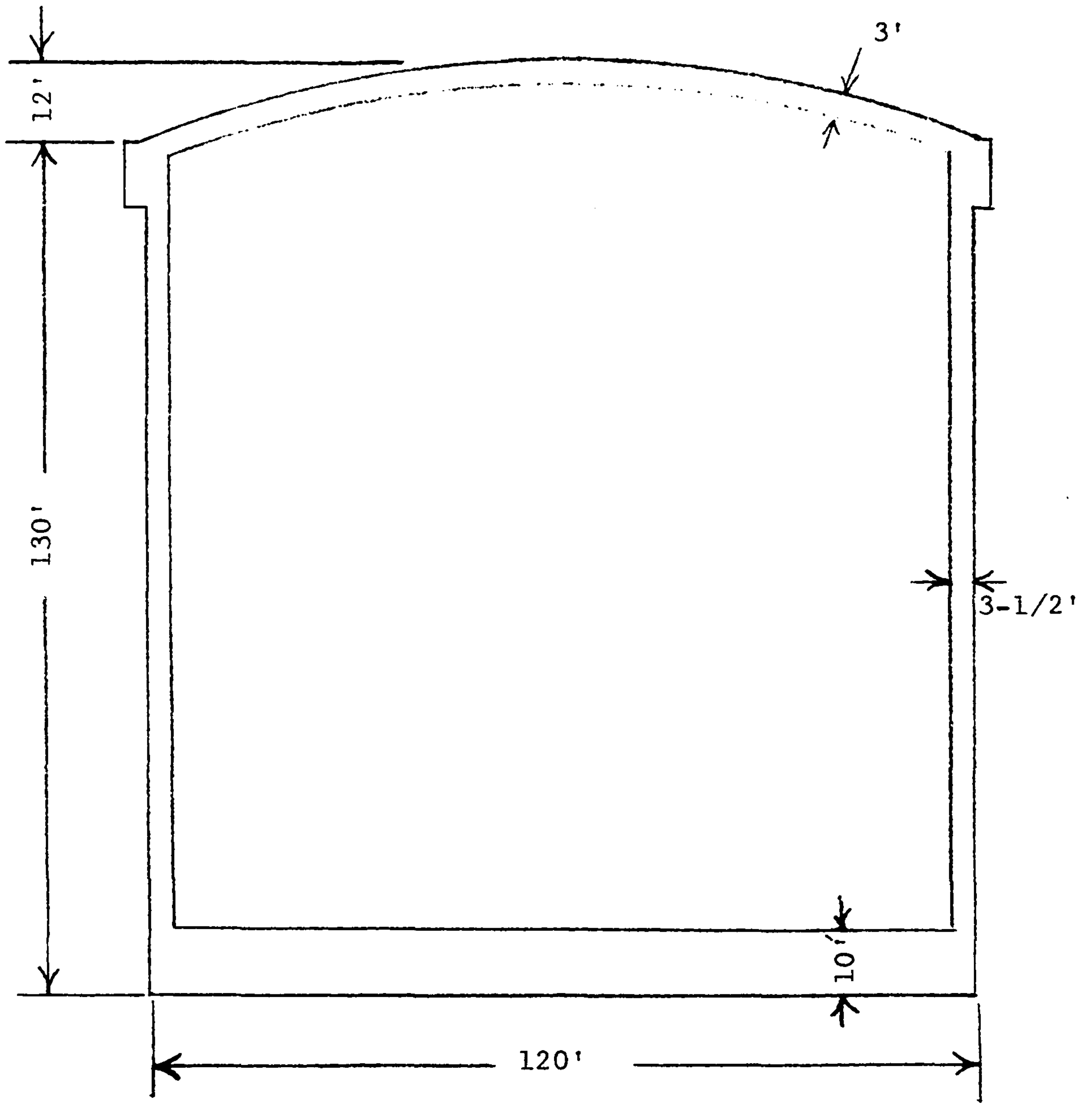


Figure 1. Type I containment structure

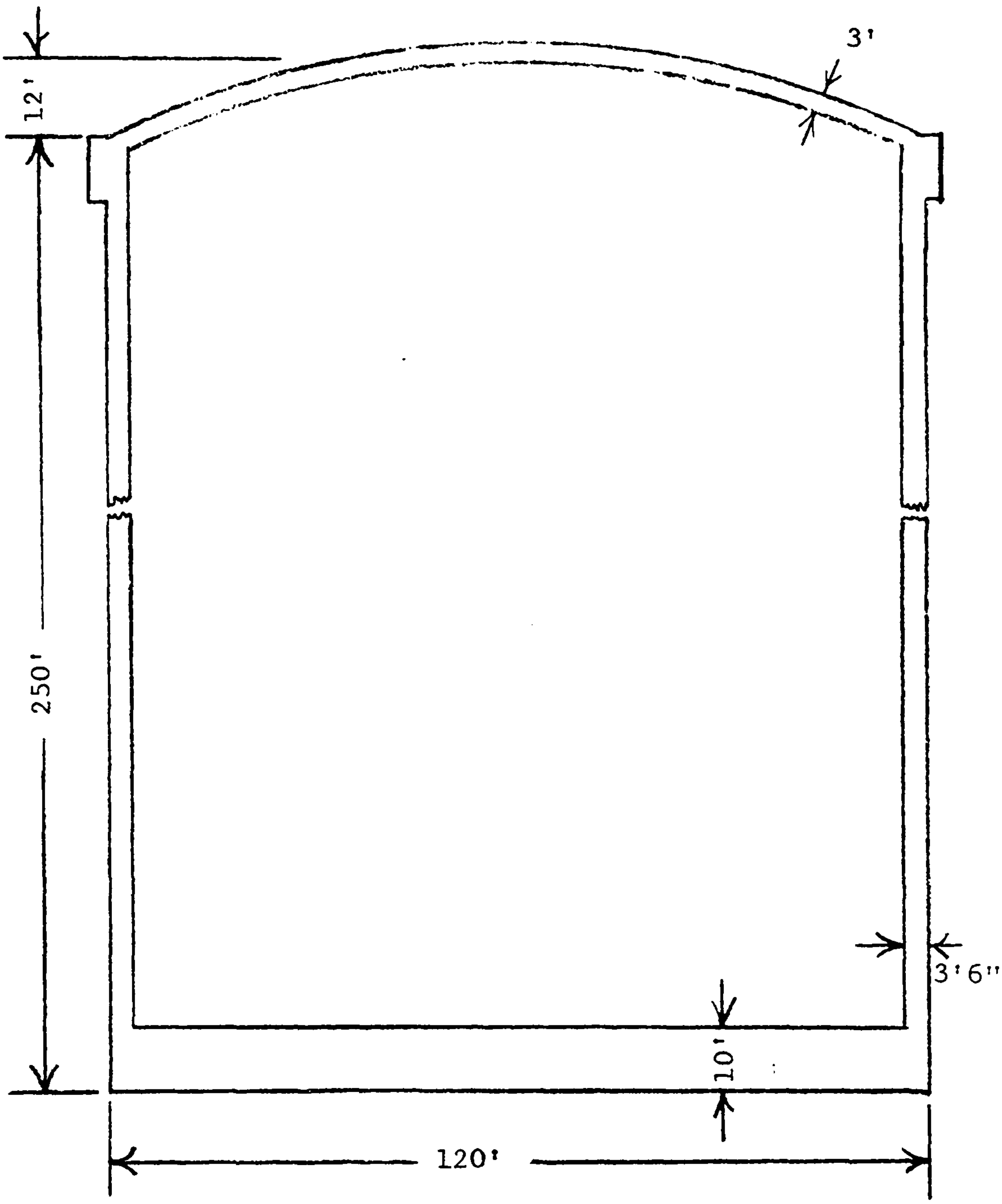


Figure 2. Type II containment structure

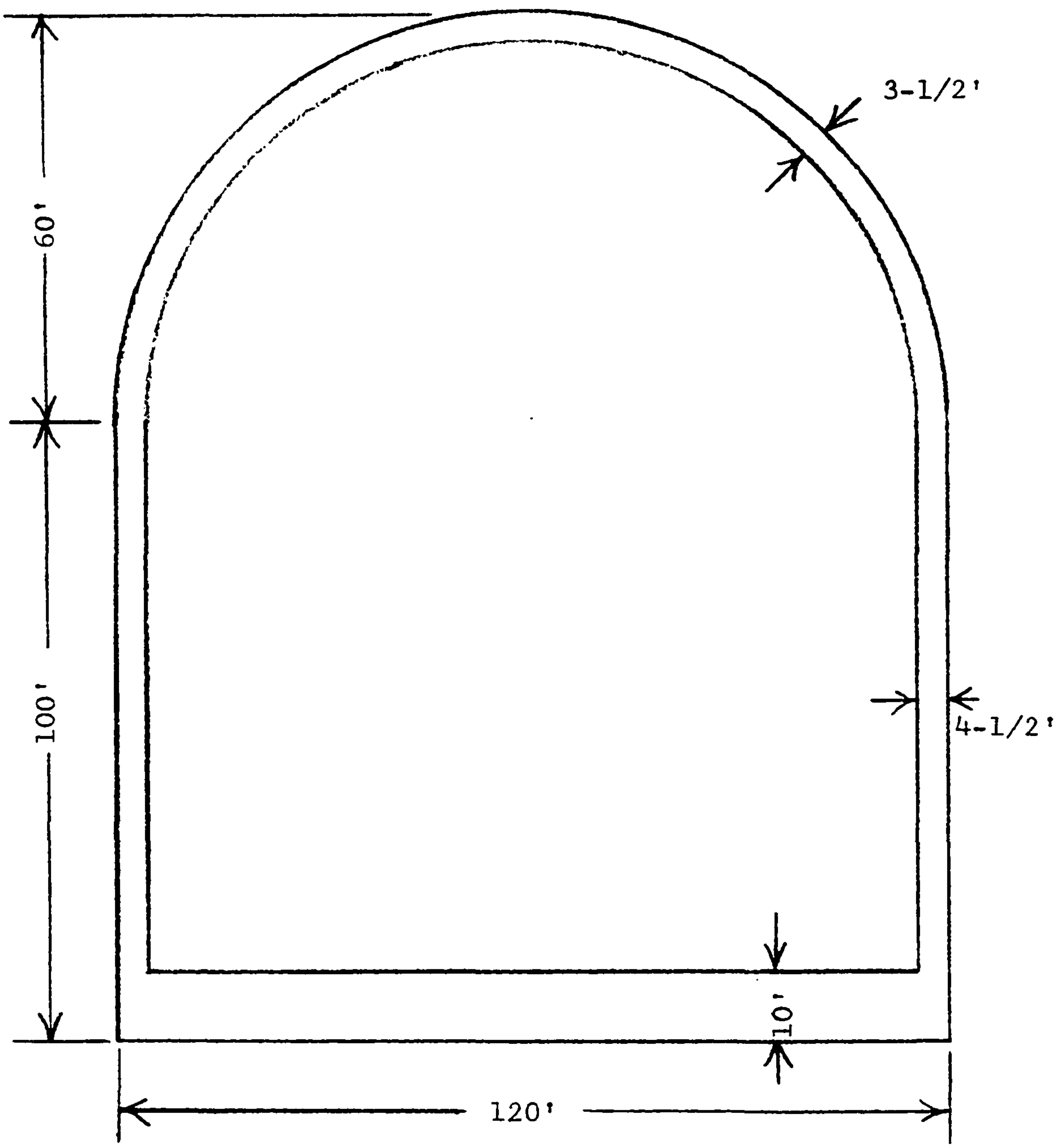


Figure 3. Type III containment structure

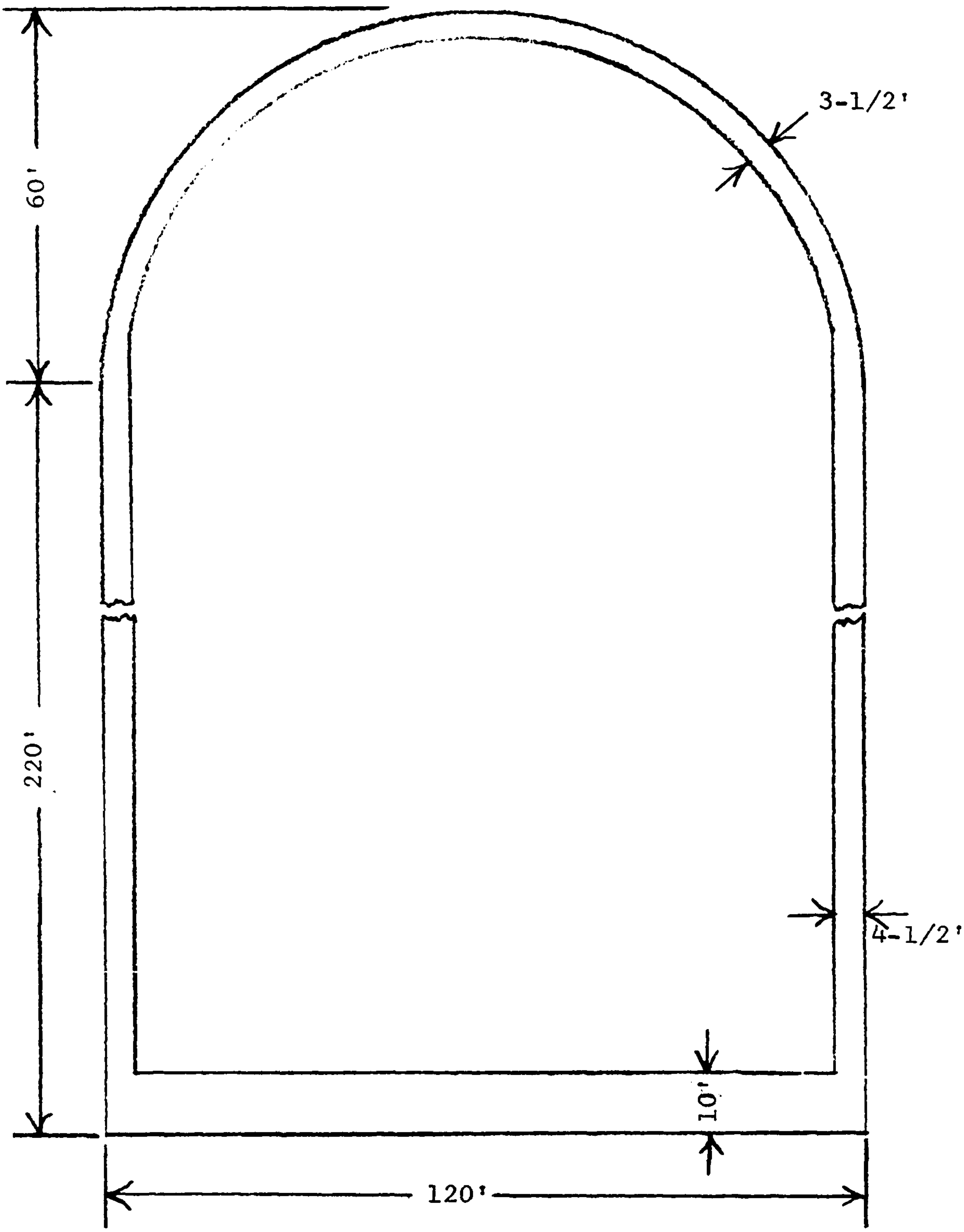


Figure 4. Type IV containment structure

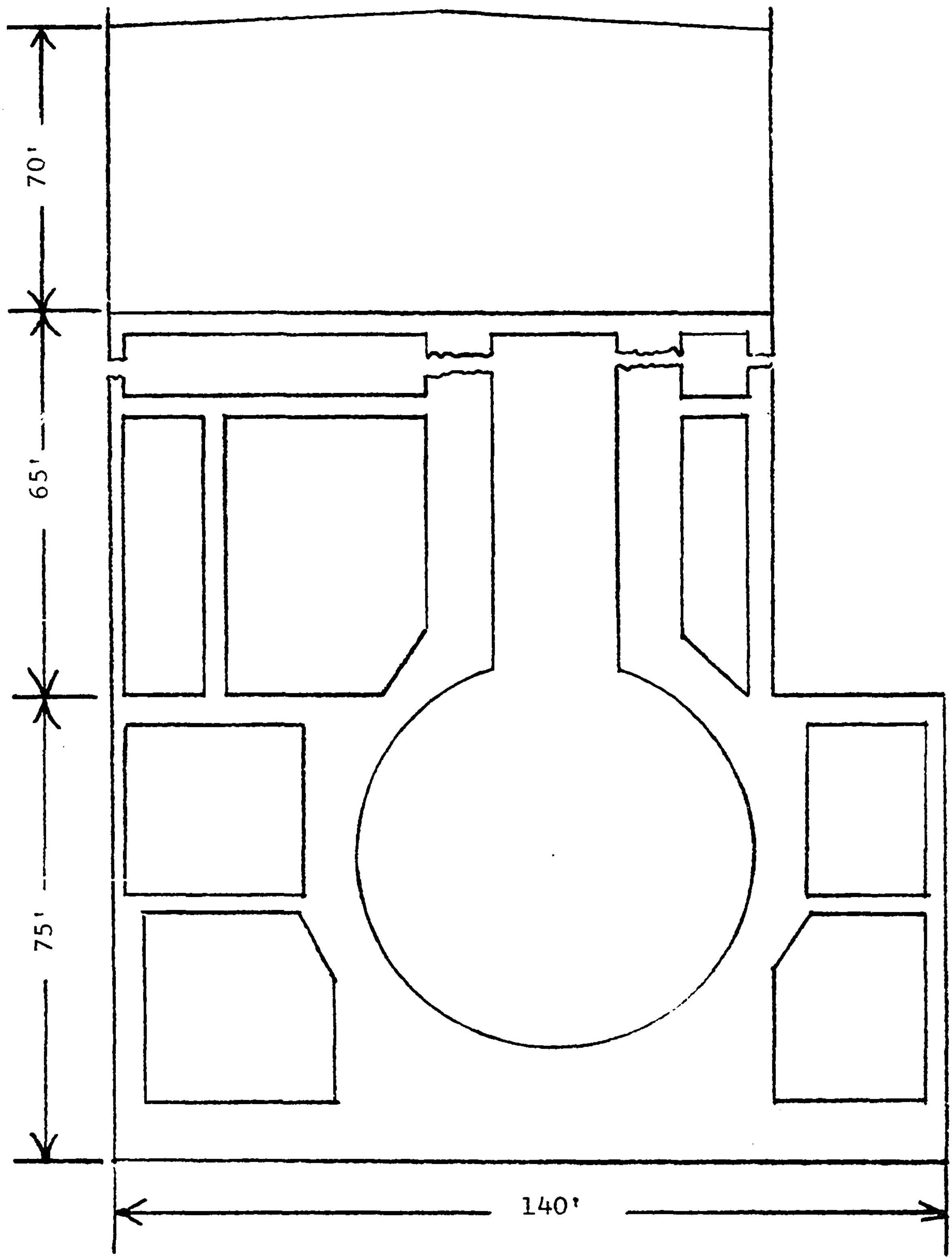


Figure 5. Type V containment structure

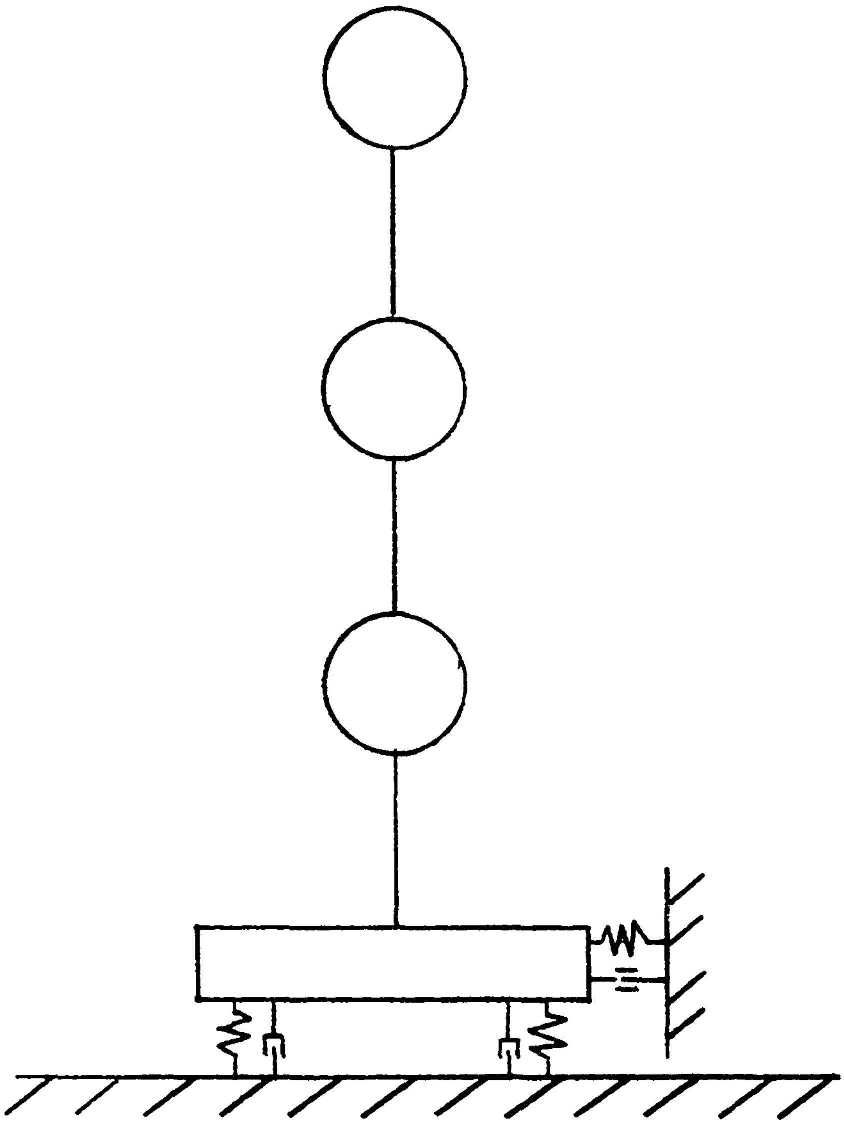


Figure 6. Typical lumped mass idealization of a containment structure

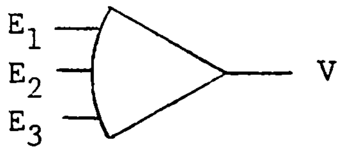
II. NOTATION AND SYMBOLS

A. Notation

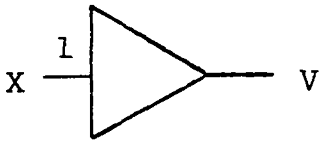
$M_1, M_2 \ldots M_n$	Lumped masses of lumped parameter system (Kip-sec^2/ft)
$K_2, K_3 \ldots K_n$	Equivalent translational stiffness of structural elements (Kip/ft)
K_1	Translational stiffness of foundation (Kip/ft)
K_r	Rotational stiffness of foundation (Kip/radian)
$C_2, C_3 \ldots C_n$	Translational energy dissipation factor in structural elements as equivalent viscous damping (Kip/ft/sec)
C_1	Translational energy dissipation factor in foundation as equivalent viscous damping (Kip/ft/sec)
C_r	Rotational energy dissipation factor in foundation as equivalent viscous damping (Kip/rad/sec)
f	Fraction of critical damping, C_{cr}, present in system (nondimensional)
M	Mass matrix (Kip-sec^2/ft)
C	Damping coefficient matrix (Kip-sec/ft)
K	Stiffness coefficient matrix
F	Inertia matrix (Kip-sec^2-ft)
L	Length (ft)
E	Young's Modulus (Kip/ft^2)
G	Shearing modulus (Kip/ft^2)
I	Moment of inertia (Kip-sec^2-ft); identity matrix

A	Effective shear area (ft^2)
θ	General rotational coordinate (radians)
X_i	General translational coordinate (ft)
ω_i	Modal circular frequency, ith mode (radians/sec)
Q	Unit vector matrix
e_i	Unit vector; ith vector
$\lceil \; \rfloor$	Matrix of modal circular frequencies (diagonal) (radian/sec)
D	Diagonal matrix of modal participation damping coefficients
P	Participation factor vector
$\{ \}$	Column vector written in text as a row vector
k	Flexural stiffness coefficient
γ, β	Proportionality factors
. . .	Zero element locations in generalized matrix
***	Non-zero element locations in generalized matrix
λ_i	Natural frequency of the ith vibrational mode
M_i	Mass of internal equipment and structures
I_i	Moment of inertia of internal equipment and structures

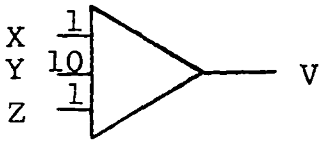
B. Symbols



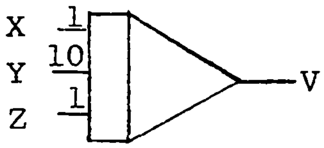
High Gain Amplifier



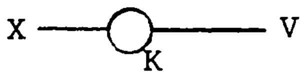
Inverter



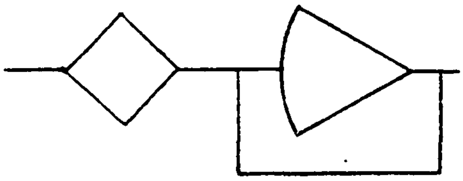
Summer



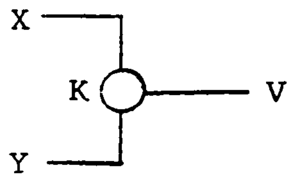
Integrator



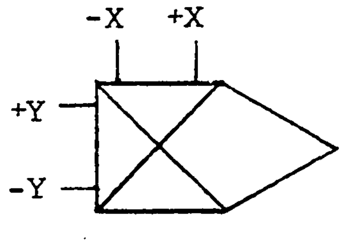
Potentiometer (Grounded)



Electronic Switch

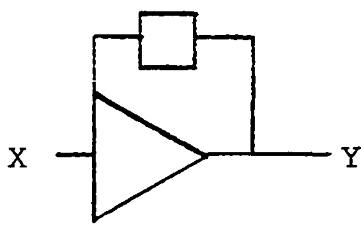


Potentiometer (Ungrounded)



Multiplier

Manual Switch



Feedback Limiter

Reference Voltage

Recorder Input

III. PROGRAM OF INVESTIGATION

A. General

In section V the containment structure and its associated foundation are idealized as a lumped parameter mathematical model and the applicable set of ordinary differential equations is developed. In section VI the basis in back of the selection of specific parameter ranges for the constants in these equations is presented. In section VII the equation set developed in section V is put into a form suitable for analog computer analysis and programed for the analog computer. Finally, in section VIII, the analytical solution method for the stiffness investigation is outlined, the procedure is applied to a particular case as an example, and a digital computer program is prepared to automate the solution procedure.

In accordance with the object of this investigation as stated in the introduction the computational tools described in the previous paragraph are applied in a parameter study of the influence of foundation stiffness and damping. A series of five nuclear containment structures (see figures one through five), to represent the range of sizes of such structures now being put into service was selected and structural parameters suitable for them were calculated from available data. Parameter runs as described in this section were made. The results of these parameter runs are discussed in section IX. The detailed input data used for the runs is tabulated in Appendix B.

B. Stiffness Investigation

The investigation of the influence of foundation stiffness on structural response is by means of digital computation. A range of foundation stiffnesses from some considerably softer than normally considered for construction to stiffnesses approaching bed rock conditions have been selected. Five general foundation situations have been studied.

The first situation is one in which foundation freedom is solely translational. A structure with this type of foundation is, in this study, designated as a "translational" structure. A second general foundation situation is one where the foundation is free to rotate but is restrained against translation. A structure with such a foundation is termed for the purposes of this study as a "rotational" structure. A third possible foundation situation, of course, is one where the foundation is both free to rotate and translate. Subcases of structures that lie in the free to translate and rotate category are structures free to translate and rotate a) where the translational and rotational stiffnesses (K_l and K_r) are proportional and b) structures where the foundation may take on any number of translational stiffness values but can assume only one rotational stiffness value. A structure with the former situation as the foundation situation is termed a "combined variable K_r" structure whereas one with the latter foundation situation is termed a "combined constant K_r" structure. For each of these situations, type I and type II containment structures were investigated for four different partitionings of base and top masses through a range of fourteen stiffness values. The type III, type IV and type V structures were investigated for all the same situations. The type III, IV and V structures were investigated using only one assumed mass distribution, however. The total investigation encompassed a total of 784 separate undamped modal analyses. For all analyses a modal absolute sum combination was used in conjunction with an idealized El Centro response spectra (see figure thirteen).

An actual foundation that could be considered as idealized by the first situation would be, in specific instances, one where the containment base is founded on a pile foundation with piling driven to bed rock or to refusal in an extremely stiff substrata. A physical situation that could be considered as idealized by situation two includes a case where the containment is on a soil cushion but yet keyed into bed rock, perhaps by a reactor vessel sump. The third general situation could idealize the physical situation where the containment rests on granular material. Lastly, a system of limited rotational restraint might be representative of a more typical pile foundation condition.

C. Damping Investigation

The investigation of foundation damping effects has been primarily by analog computer methods. However, for comparative purposes, a complete set of modal absolute sum calculations with varied modal damping percentages has also been made. From the parametric study of stiffness influence it was observed that the type III and type IV reinforced concrete structures were very similar in structure response to their type I and II prestressed concrete counterparts. The type V structure fell, response wise, also in the general behavior pattern of type I and type II structures. In view of these facts it was considered tha[illegible]tailed investigation of type I and type II structure found[illegible] damping influence would adequately cover the range of behavior. The damping investigation was, as a result, restricted to an investigation of these two cases.

For each of these two structural types, the situations where a) the foundation is free only to translate, b) free only to rotate and c) where both translational and rotational freedom are present have been investigated. For each of these situations a set of twelve separate foundation stiffnesses (twelve of the fourteen stiffnesses previously employed in the stiffness investigation) have been applied. For each foundation stiffness value the accompanying damping percentage was varied from zero to forty percent in six increments. All in all, two hundred sixteen cases were investigated for each structure to sum to a grand total of four hundred thirty-two separate analog computer runs.

IV. REVIEW OF RELATED LITERATURE

The influence of foundation conditions on the performance of structures undergoing ground motions has for some time been a matter of speculation by engineers. As early as 1930 Jacobsen (20) studied the cantilever beam embedded in an elastic material. Biot (4) in his engineering seismology paper obtained a simple expression to represent the elastic stiffness coefficient for the rocking motion of an infinite strip resting on an elastic half space. Merritt and Housner (26) investigated the effect of rocking on the maximum base shear force and fundamental period of tall buildings under typical earthquake ground motion. Their study was restricted to flexible structures with spread footings. The influence of foundation conditions on the period of framed structures was also investigated by Salvadori and Heer (36) for buildings of curtain wall type. They used the elastic stiffness coefficient of Biot and a horizontal shear modulus in developing a period that included elastic foundation considerations. Thomson (41) extended Merritt and Housner's study analytically to include consideration of the more generalized case of a tall, flexible structure with spread footings. The studies by both Merritt and Housner and Thomson supported a conclusion that, in considering structures of a tall, flexible type, neglect of foundation properties was justifiable and such structures could, without loss of accuracy, be analyzed as fixed base structures.

Field observations indicate, however, that the above conclusions may not be generalized to include stiff, shear wall type structures. Surveys of earthquake damage in Japan (23), where structures are as a practice rather short and of shear wall construction, have indicated significant influence by foundation conditions on structural performance. American experience with earthquake response of stiff structures has also led American engineers (25) to believe that foundation conditions can have considerable effect on structural response.

One observation of structural response as influenced by foundation conditions that is particularly relevant to this study is Housner's (14) response spectrum from the Arvin-Tehachapi earthquake. For this earthquake Housner developed response spectra from an accelogram record obtained from the basement of a large (141 feet high by 51 feet wide by 217 feet long) monolithic reinforced concrete building and from an accelogram record of a ground station in the close proximity of the same building. The response spectrum from the ground acceleration recorded in the structure's basement was, in the period range from zero to one and one-half seconds, approximately forty percent smaller than the spectrum developed from an accelogram obtained from a ground station in the close proximity for the East-West direction. The measured period of the structure in this direction was 0.49 seconds. It is noteworthy that containment structures have calculated periods in the 0.10 to 0.50 second range and, also, have dimensions of the same scale as Housner's structure.

Researchers at the Naval Research Laboratory have also studied the foundation-structure interaction problem starting in early 1960. Dealing primarily with naval shipboard equipment it was reported by Belsheim and O'Hara (2) that the foundation-structure interaction effect was responsible for a pronounced dip in the foundation spectrum curves near the natural frequency of the particular structure. Spectrum curves that incorporated this spectrum dip as a function of equipment weight were incorporated into the dynamic design method for major shipboard equipment (30).

Also related to the foundation-structure interaction problem has been the response spectrum development work of Housner. Housner averaged spectrum values from selected earthquakes at various locations in California after making adjustments for shock magnitude and epicenter distance. These averaged curves have been useful in design. However, in practice, magnitudes taken from such curves can be conservative or non-conservative depending on the degree of structure-foundation interaction.

Rosenblueth (35) in 1961 pointed out the potential significance of foundation-structure interaction in commenting to the effect that there were indications that spectra computed from free ground motion greatly overestimated structural response in specific natural period ranges.

The effect of structure-foundation interaction has also been studied by Lycan and Newmark (24). In their study the effect of soil inertia was found to have a significant effect on structural response.

Recently, Scarvazzo (37) has analytically coupled an N-mass structure with a one dimensional ground wave and reduced the problem to that of the solution of a Volterra type integral equation. For some simple free ground acceleration functions he has shown reductions in spectral acceleration to be significant. Also very recently Jennings and Kuroiwa (21) have shown that even for firm foundations, where fairly stiff shear walled structures are involved, foundation interaction is measurable.

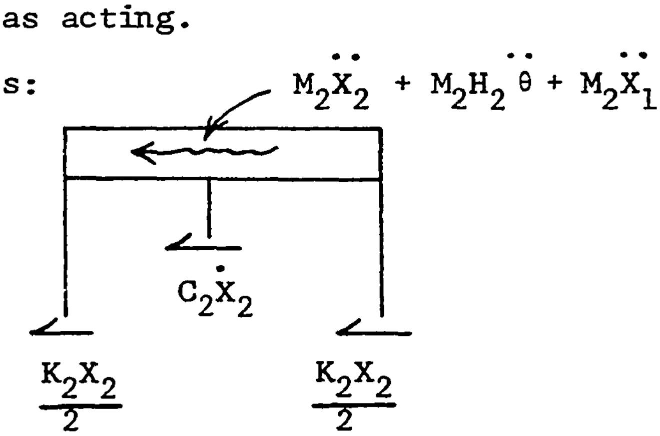
Lastly, Seed and Idriss (39)(40) have very recently shown lumped masses to be an appropriate foundation representation when ground surface, rock surface and soil layer boundaries are essentially horizontal.

V. MATHEMATICAL FORMULATION OF PROBLEM

A. General Two Free Mass System

The problem of a structure undergoing ground motion can be reduced with little loss of accuracy to that of a lumped parameter system of effective masses connected together by equivalent stiffness springs. Normal modeling of a containment system would include the assumption of sufficient lumped masses so that the dynamics of the lumped mass system closely approximated that of the distributed mass structure. For this investigation the details of mass distribution and structural stiffness are not too important, however, and the structure can be described with sufficient accuracy by using a two free mass system. One mass is taken to describe the base mass and contributing side wall mass. A second mass is given a value appropriate to describe the dome and upper side wall contribution. The model to represent this situation is shown as figure seven. In order to derive the differential equations of motion each mass is isolated, in turn, as a free body and the following forces are observed as acting.

For the top mass:



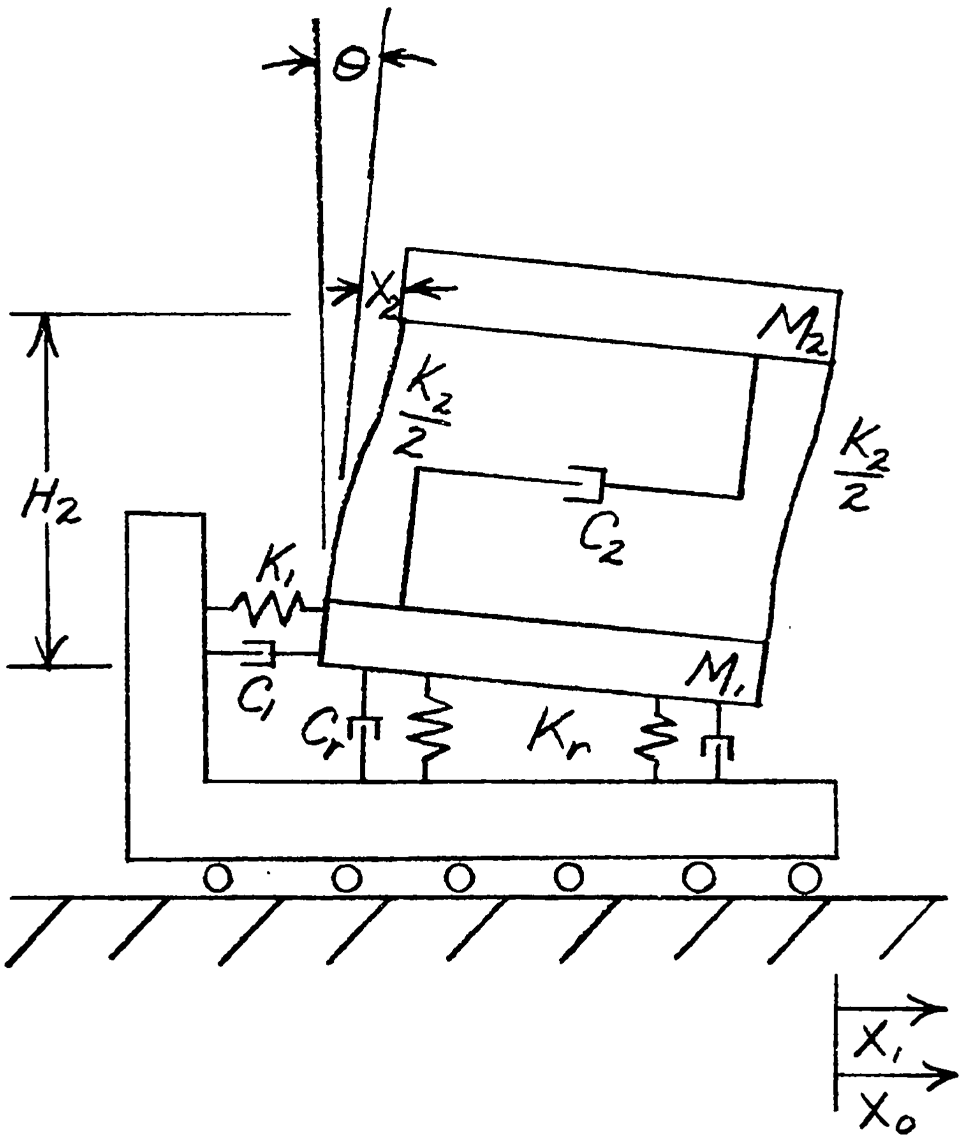
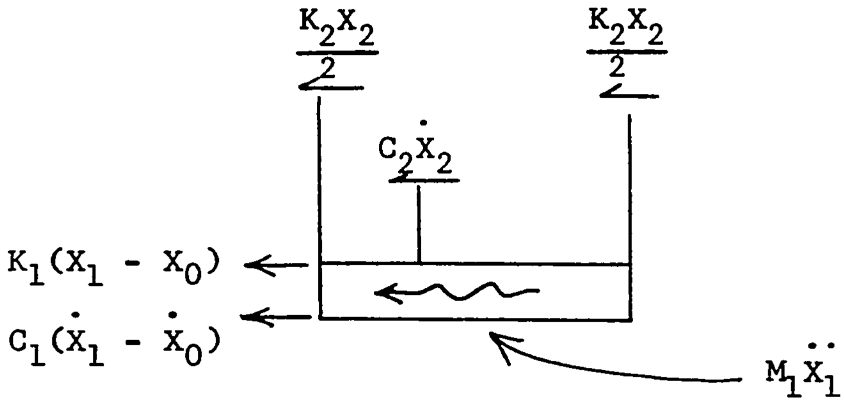


Figure 7. Idealization of generalized two free mass problem

Likewise, for the base mass:

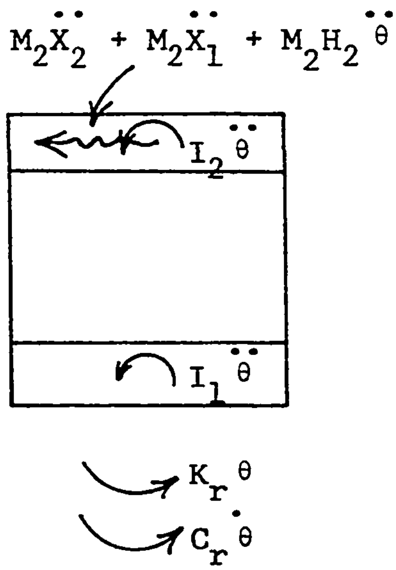


Summing forces in the horizontal direction after applying D'Alembert's principle yields the following equations of motion.

$$M_2\ddot{X}_2 + M_2H_2\ddot{\theta} + M_2\ddot{X}_1 + C_2\dot{X}_2 + K_2X_2 = 0 \qquad (5.1)$$

$$M_1\ddot{X}_1 + C_1\dot{X}_1 + K_1X_1 - C_2\dot{X}_2 - K_2X_2 - C_1\dot{X}_0 - K_1X_0 = 0$$

Likewise, considering the system as a unit and considering rotational equilibrium about the centroid of the base the following forces are shown as acting:



Taking moments about the base after applying D'Alembert's principle yields

$$M_2H_2\ddot{X}_2 + M_2H_2\ddot{X}_1 + M_2H_2^2\ddot{\theta} + (I_1 + I_2)\ddot{\theta} + C_r\dot{\theta} + K_r\theta = 0 \tag{5.2}$$

Letting $I_0 = I_1 + I_2$ and rearranging the preceding three equations into matrix form yields

$$\begin{bmatrix} M_2 & M_2 & M_2H_2 \\ 0 & M_1 & 0 \\ M_2H_2 & M_2H_2 & I_0 + M_2H_2^2 \end{bmatrix} \begin{bmatrix} \ddot{X}_2 \\ \ddot{X}_1 \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} C_2 & 0 & 0 \\ -C_2 & C_1 & 0 \\ 0 & 0 & C_r \end{bmatrix} \begin{bmatrix} \dot{X}_2 \\ \dot{X}_1 \\ \dot{\theta} \end{bmatrix} \tag{5.3}$$

$$\begin{bmatrix} K_2 & 0 & 0 \\ -K_2 & K_1 & 0 \\ 0 & 0 & K_r \end{bmatrix} \begin{bmatrix} X_2 \\ X_1 \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ C_1\dot{X}_0 + K_1X_0 \\ 0 \end{bmatrix}$$

It can be seen that the preceding formulation has lead to non symmetric off diagonal elements in the system matrices. This is undesirable. To symmetrize the matrices one first defines as new variable X^1 as follows:

$$X_1^{\,1} = X_1 - X_0$$

Correspondingly,

$$\dot{X}^1 = \dot{X} - \dot{X}_o$$

$$\ddot{X}^1 = \ddot{X} - \ddot{X}_0$$

Making these substitutions yields

$$M_2\ddot{X}_2 + M_2H_2\ddot{\theta} + M_2(\ddot{X}_1{}^1 + \ddot{X}_0) + C_2\dot{X}_2 + K_2X_2 = 0 \tag{5.4}$$

$$M_1(\ddot{X}_1{}^1 + \ddot{X}_0) + C_1(\dot{X}^1 + \dot{X}_0) + K_1(X_1{}^1 + X_0) - C_2\dot{X}_2$$

$$-K_2X_2 - C_1\dot{X}_0 - K_1\dot{X}_0 = 0$$

$$M_2H_2\ddot{X}_2 + M_2H_2(\ddot{X}_1{}^1 + \ddot{X}_0) + M_2H_2{}^2\ddot{\theta} + I_0\ddot{\theta} + C_r\dot{\theta} + K_r\theta = 0.$$

Rearranging

$$M_2\ddot{X}_2 + M_2\ddot{X}_1{}^1 + M_2H_2\ddot{\theta} + C_2\dot{X}_2 + K_2X_2 = -M_2\ddot{X}_0 \tag{5.5}$$

$$M_1\ddot{X}_1{}^1 + C_1\dot{X}_1{}^1 + K_1X_1{}^1 - C_2\dot{X}_2 - K_2X_2 = -M_1\ddot{X}_0$$

$$M_2H_2\ddot{X}_2 + M_2H_2\ddot{X}_1{}^1 + M_2H_2{}^2\ddot{\theta} + I_0\ddot{\theta} + C_r\dot{\theta} + K_r\theta = -M_2H_2\ddot{X}_0$$

Now, making the additional linear transformation

$$X_2{}^1 = X_2 + X_1{}^1$$

And substituting into the previous equations yields

$$M_2(\ddot{X}_2 - \ddot{X}_1{}^1) + M_2\ddot{X}_1{}^1 + M_2H_2\ddot{\theta} + C_2(\dot{X}_2{}^1 - \dot{X}_1{}^1) \tag{5.6}$$

$$+K_2(X_2{}^1 - X_1{}^1) = -M_2\ddot{X}_0$$

$$M_1\ddot{X}_1{}^1 + C_1\dot{X}_1{}^1 + K_1X_1{}^1 - C_2(\dot{X}_2{}^1 - \dot{X}_1{}^1) - K_2(X_2{}^1 - X_1{}^1)$$

$$= -M_1\ddot{X}_0$$

$$M_2H_2(\ddot{X}_2{}^1 - \ddot{X}_1{}^1) + M_2H_2\ddot{X}_1{}^1 + M_2H_2{}^2\ddot{\theta} + I_0\ddot{\theta} + C_r\dot{\theta} + K_r\theta$$

$$= -M_2H_2\ddot{X}_0$$

Again, rearranging,

$$M_2\ddot{X}_2^{\prime} + M_2H_2\ddot{\theta} + C_2\dot{X}_2^{\prime} - C_2\dot{X}_1^{\prime} + K_2X_2^{\prime} - K_2X_1^{\prime} = -M_2\ddot{X}_0 \qquad (5.7)$$

$$M_1\ddot{X}_1^{\prime} + C_1\dot{X}_1^{\prime} + K_1X_1^{\prime} - C_2\dot{X}_2^{\prime} + C_2\dot{X}_1^{\prime} - K_2X_2^{\prime} + K_2X_1^{\prime} = -M_1\ddot{X}_0$$

$$M_2H_2\ddot{X}_2^{\prime} + M_2H_2^2\ddot{\theta} + I_0\ddot{\theta} + C_r\dot{\theta} + K_r\theta = -M_2H_2\ddot{X}_0$$

Placing into matrix form and dropping primes for mathematical simplicity yields

$$\begin{bmatrix} M_2 & 0 & M_2H_2 \\ 0 & M_1 & 0 \\ M_2H_2 & 0 & I_0 + M_2H_2^2 \end{bmatrix} \begin{bmatrix} \ddot{X}_2 \\ \ddot{X}_1 \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} C_2 & -C_2 & 0 \\ -C_2 & C_1+C_2 & 0 \\ 0 & 0 & C_r \end{bmatrix} \begin{bmatrix} \dot{X}_2 \\ \dot{X}_1 \\ \dot{\theta} \end{bmatrix}$$

$$+ \begin{bmatrix} K_2 & -K_2 & 0 \\ -K_2 & K_1+K_2 & 0 \\ 0 & 0 & K_r \end{bmatrix} \begin{bmatrix} X_2 \\ X_1 \\ \theta \end{bmatrix} = \begin{bmatrix} -M_2\ddot{X}_0 \\ -M_1\ddot{X}_0 \\ -M_2H_2\ddot{X}_0 \end{bmatrix} \qquad (5.8)$$

B. General Three Free Mass System

The general three mass development proceeds in a manner analogous to that used in the previous section for the two free mass system. It is incorporated in this study, however, inasmuch as a comparison of the resultant form of the derived equations with those arrived at in the preceding case permits an immediate generalization to the general multimass case. The model representing the system is shown in figure eight. Proceeding as previously, each mass is isolated in turn as a free body. The following forces can then be observed as acting.

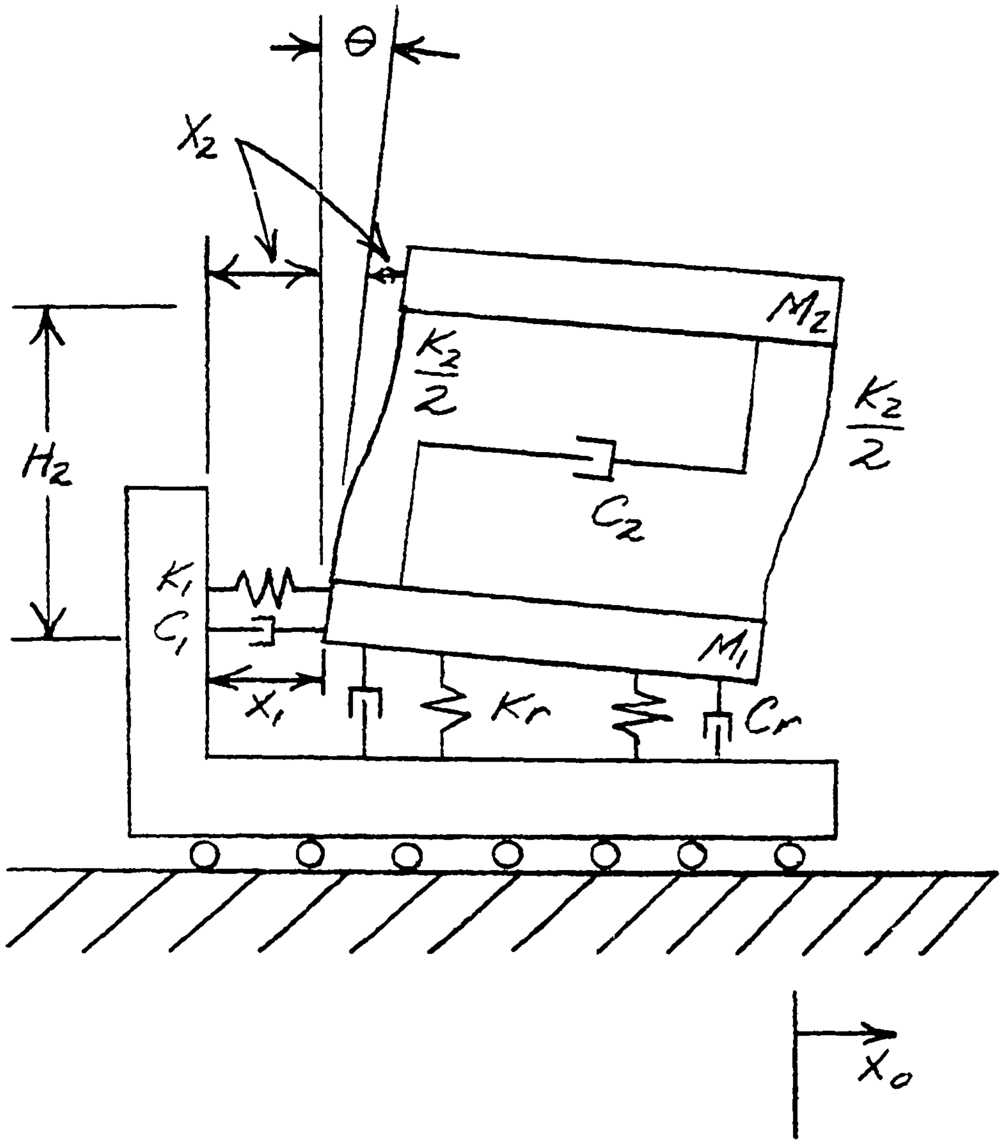
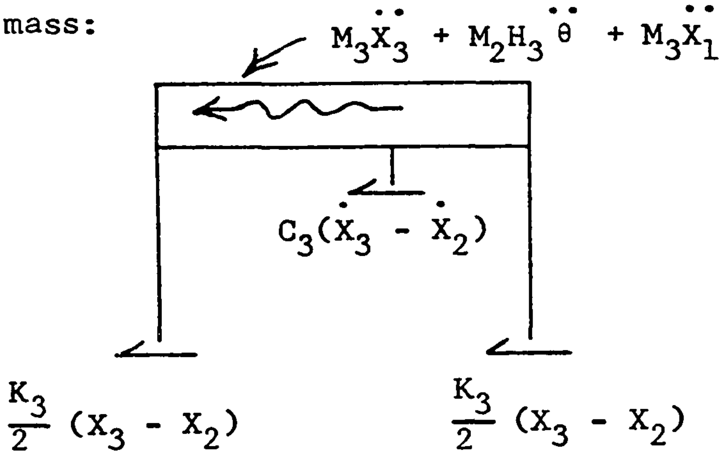
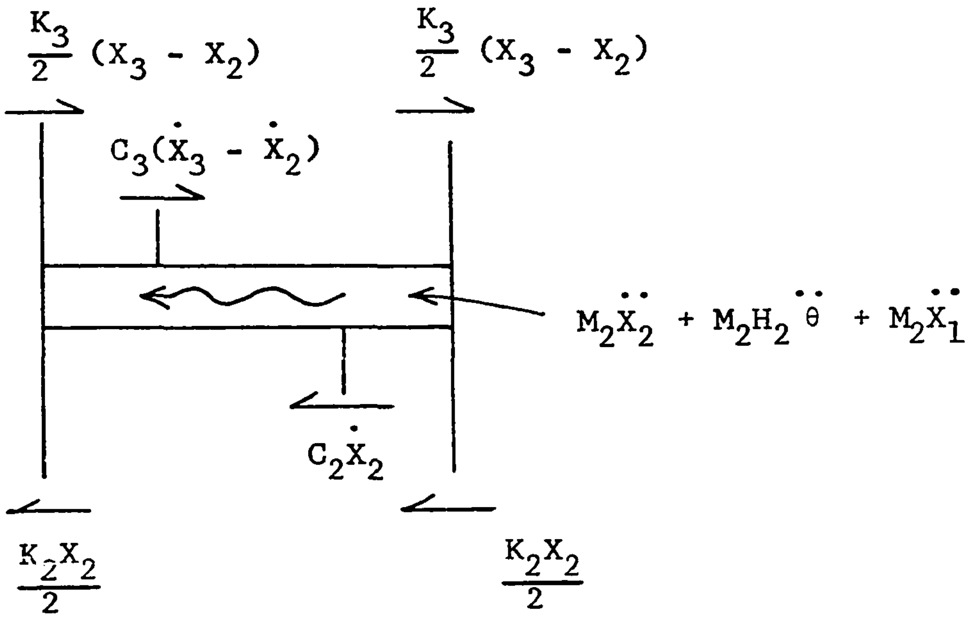


Figure 7a. Idealization of generalized two free mass in final transformed coordinates

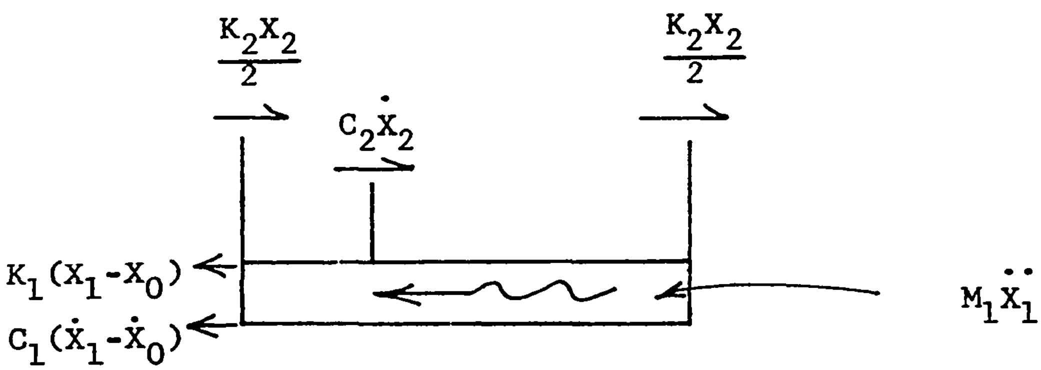
For the top mass:



For the middle mass:



Likewise, for the first mass



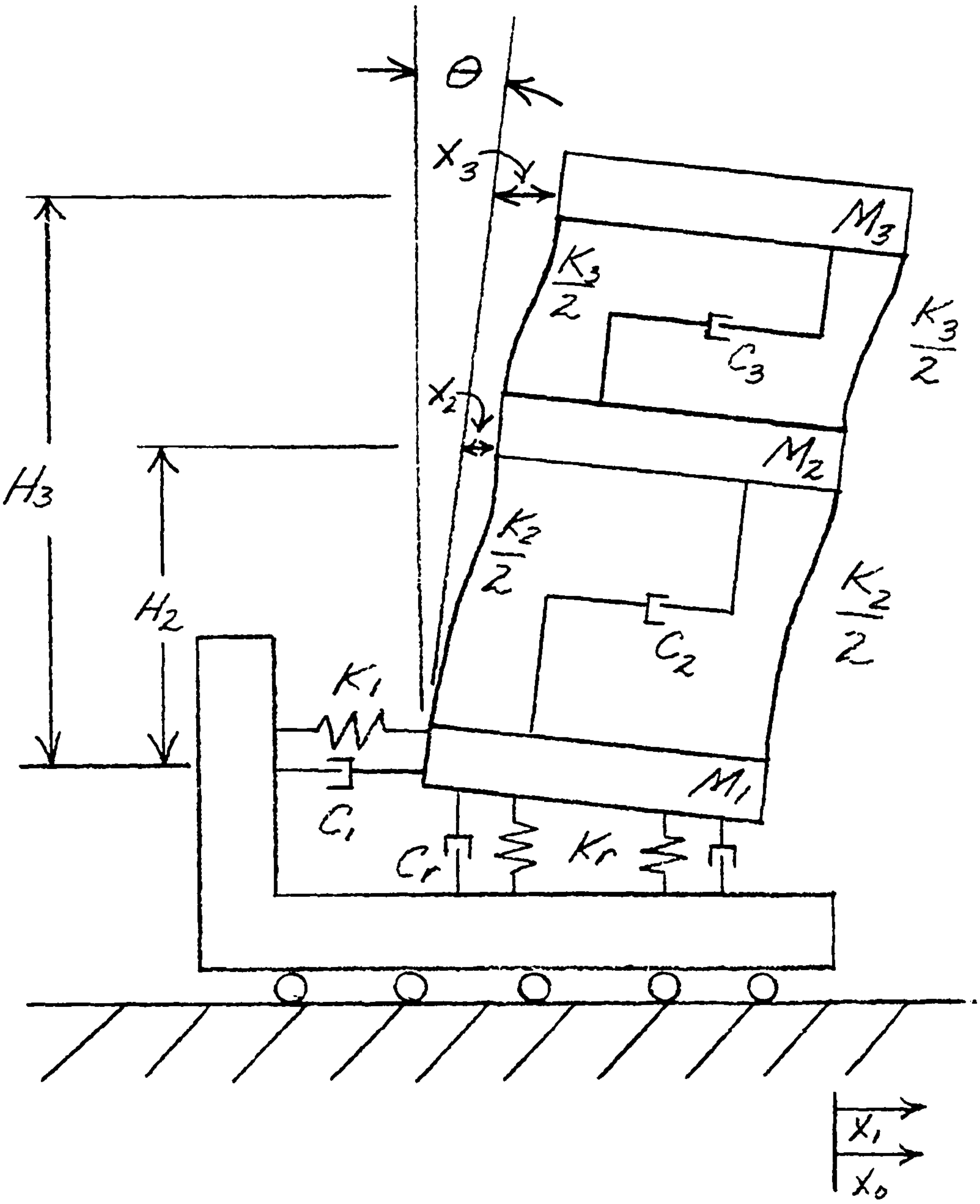


Figure 8. Idealization of generalized three mass system

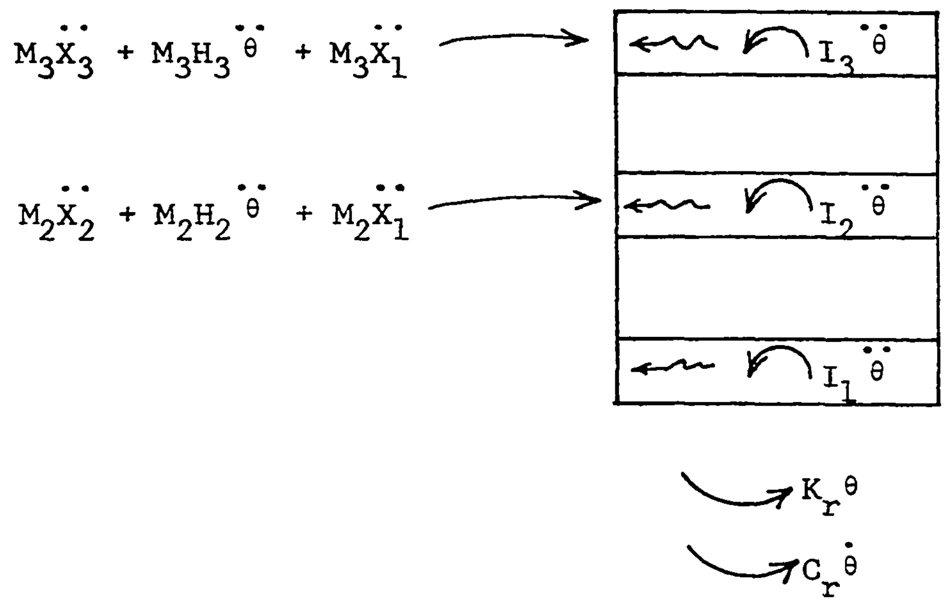
Summing forces in the horizontal direction after applying D'Alembert's principle yields the following set of equations of motion.

$$M_3\ddot{X}_1 + M_3\ddot{X}_3 + M_3H_3\ddot{\theta} + C_3\dot{X}_3 + K_3X_3 - C_3\dot{X}_2 - K_3X_2 = 0 \qquad (5.9)$$

$$M_2\ddot{X}_1 + M_2\ddot{X}_2 + M_2H_2\ddot{\theta} + C_2\dot{X}_2 + K_2X_2 - C_3\dot{X}_3 - K_3X_3$$
$$+C_3\dot{X}_2 + K_3X_2 = 0$$

$$M_1\ddot{X}_1 + C_1\dot{X}_1 + K_1X_1 - C_2\dot{X}_2 - K_2X_2 - C_1\dot{X}_0 - K_1X_0 = 0$$

Again considering the system as a unit and investigating rotational equilibrium around the centroid of the base mass shows the following forces as acting.



Taking moments around the centroid of the base after applying D'Alembert's principle yields

$$M_3\ddot{X}_3H_3 + M_2\ddot{X}_2H_2 + M_3\ddot{X}_1H_2 + M_3H_3^{2}\ddot{\theta} + M_2H_2^{2}\ddot{\theta} + (I_1 + I_2 + I_3)\ddot{\theta} + C_r\dot{\theta} + K_r\theta = 0 \tag{5.10}$$

Letting $I_0 = I_1 + I_2 + I_3$ and rearranging the preceding set of four equations into matrix form yields:

$$\begin{bmatrix} M_3 & 0 & M_3 & M_3H_3 \\ 0 & M_2 & M_2 & M_2H_2 \\ 0 & 0 & M_1 & 0 \\ M_3H_3 & M_2H_2 & M_3H_3 + M_2H_2 & I_0 + M_3H_3^2 + M_2H_2^2 \end{bmatrix} \begin{bmatrix} \ddot{X}_3 \\ \ddot{X}_2 \\ \ddot{X}_1 \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} C_3 & -C_3 & 0 & 0 \\ -C_3 & C_2 + C_3 & 0 & 0 \\ 0 & -C_2 & C_1 & 0 \\ 0 & 0 & 0 & C_r \end{bmatrix} \begin{bmatrix} \dot{X}_3 \\ \dot{X}_2 \\ \dot{X}_1 \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} K_3 & -K_3 & 0 & 0 \\ -K_3 & K_2 + K_3 & 0 & 0 \\ 0 & -K_2 & K_1 & 0 \\ 0 & 0 & 0 & K_r \end{bmatrix} \begin{bmatrix} X_3 \\ X_2 \\ X_1 \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ C_1\dot{X}_0 + K_1X_0 \\ 0 \end{bmatrix} \tag{5.11}$$

Likewise, if the first mass movement is expressed in reference to the translating foundation instead of in reference to a fixed location the equations of motion become:

$$
\begin{aligned}
&M_3\ddot{X}_3 + M_3H_3\ddot{\theta} + M_3\ddot{X}_1 + M_3\ddot{X}_0 + C_3\dot{X}_3 + K_3X_3 - C_3\dot{X}_2 - K_3X_2 = 0 \\
&+M_2\ddot{X}_2 + M_2H_2\ddot{\theta} + M_2\ddot{X}_1 + M_2\ddot{X}_0 + C_2\dot{X}_2 + K_2X_2 - C_3\dot{X}_3 - K_3X_3 \\
&+C_3\dot{X}_2 + K_3X_2 = 0 \qquad (5.12) \\
&M_1\ddot{X}_1 + M_1\ddot{X}_0 + C_1\dot{X}_1 + K_1X_1 - C_2\dot{X}_2 - K_2X_2 = 0 \\
&M_3\ddot{X}_3H_3 + M_3\ddot{X}_1H_3 + M_3\ddot{X}_0H_3 + M_2\ddot{X}_2H_2 + M_2\ddot{X}_1H_2 + M_2\ddot{X}_0H_2 \\
&+M_3H_3^{2\,\cdot\cdot} + M_2H_2^{2}\ddot{\theta} + I_0\ddot{\theta} + C_r\dot{\theta} + K_r\theta = 0
\end{aligned}
$$

Again rearranging into matrix form these equations can be expressed as follows:

$$
\begin{bmatrix}
M_3 & 0 & M_3 & M_3H_3 \\
0 & M_2 & M_2 & M_2H_2 \\
0 & 0 & M_1 & 0 \\
M_3H_3 & M_2H_2 & M_3H_3 + M_2H_2 & I_0 + M_3H_3^2 + M_2H_2^2
\end{bmatrix}
\begin{bmatrix}
\ddot{X}_3 \\ \ddot{X}_2 \\ \ddot{X}_1 \\ \ddot{\theta}
\end{bmatrix}
\qquad (5.13)
$$

$$
+
\begin{bmatrix}
C_3 & -C_3 & 0 & 0 \\
-C_3 & C_2 + C_3 & 0 & 0 \\
0 & -C_2 & C_1 & 0 \\
0 & 0 & 0 & C_r
\end{bmatrix}
\begin{bmatrix}
\dot{X}_3 \\ \dot{X}_2 \\ \dot{X}_1 \\ \dot{\theta}
\end{bmatrix}
$$

$$+ \begin{bmatrix} K_3 & -K_3 & 0 & 0 \\ -K_3 & K_2 + K_3 & 0 & 0 \\ 0 & -K_2 & K_1 & 0 \\ 0 & 0 & 0 & K_r \end{bmatrix} \begin{bmatrix} X_3 \\ X_2 \\ X_1 \\ \theta \end{bmatrix} = \begin{bmatrix} -M_3\ddot{X}_0 \\ -M_2\ddot{X}_0 \\ -M_1\ddot{X}_0 \\ -(M_3H_3+M_2H_2)\ddot{X}_0 \end{bmatrix}$$

As with the two free mass development, neither of the preceding two formulations is desirable because of the lack of symmetry of the matrices involved. Therefore the further transformations, $X_3^1 = X_3 + X_1^1$ and $X_2^1 = X_2 + X_1^1$ are made. The equations of motion then become:

$$M_3\ddot{X}_3^1 + M_3H_3\ddot{\theta} + M_3\ddot{X}_0 + C_3\dot{X}_3^1 - C_3\dot{X}_1 + K_3X_3^1 - K_3X_1$$

$$-C_3\dot{X}_2^1 + C_3\dot{X}_1 - K_3X_2^1 + K_3X_1 = 0 \tag{5.14}$$

$$M_2\ddot{X}_2^1 + M_2H_2\ddot{\theta} + M_2\ddot{X}_0 + C_2\dot{X}_2^1 - C_2\dot{X}_1 + K_2X_2^1 - K_2X_1 - C_3\dot{X}_3^1$$

$$+C_3\dot{X}_1^1 - K_3X_3^1 + K_3X_1 + C_3\dot{X}_2^1 - C_3X_1^1 + K_3X_2^1 - K_3X_1 = 0$$

$$M_1\ddot{X}_1 + M_1\ddot{X}_0 + C_1\dot{X}_1 + K_1X_1 - C_2\dot{X}_2^1 + C_2\dot{X}_1 - K_2X_2^1 + K_2X_1$$

$$M_3\ddot{X}_3^1H_3 + M_2\ddot{X}_2^1H_2 + M_3\ddot{X}_0H_3 + M_2\ddot{X}_0H_2 + M_3H_3^2\ddot{\theta} + M_2H_2^2\ddot{\theta}$$

$$+ I_0 + C_r\dot{\theta} + K_r\theta = 0.$$

After eliminating like terms these equations reduce to:

$$M_3\ddot{X}_3^1 + M_3H_3\ddot{\theta} + M_3\ddot{X}_0 + C_3\dot{X}_3^1 + K_3X_3^1 - C_3\dot{X}_2^1 - K_3X_2^1 = 0$$

$$M_2\ddot{X}_2^1 + M_2H_2\ddot{\theta} + M_2\ddot{X}_0 + C_2\dot{X}_2^1 - C_2\dot{X}_1 + K_2X_2^1 - K_2X_1 \tag{5.15}$$

$$-C_3\dot{X}_3{}^1 - K_3X_3{}^1 + C_3X_2{}^1 + K_3X_2{}^1 = 0$$

$$M_1\ddot{X}_1 + M_1\ddot{X}_0 + C_1\dot{X}_1 + K_1X_1 - C_2\dot{X}_2{}^1 + C_2\dot{X}_1 - K_2X_2{}^1 + K_2X_1$$

$$M_3\ddot{X}_3{}^1H_3 + M_2\ddot{X}_2{}^1H_2 + M_3\ddot{X}_0H_3 + M_2\ddot{X}_0H_2 + M_3H_3{}^2\ddot{\theta} + M_2H_2{}^2\ddot{\theta}$$
$$+I_0\ddot{\theta} + C_r\dot{\theta} + K_r\theta = 0.$$

Placing into matrix form and dropping primes for mathematical simplicity yields:

$$\begin{bmatrix} M_3 & 0 & 0 & M_3H_3 \\ 0 & M_2 & 0 & M_2H_2 \\ 0 & 0 & M_1 & 0 \\ M_3H_3 & M_2H_2 & 0 & I_0 + M_3H_3^2 + M_2H_2^2 \end{bmatrix} \begin{bmatrix} \ddot{X}_3 \\ \ddot{X}_2 \\ \ddot{X}_1 \\ \ddot{\theta} \end{bmatrix} \tag{5.16}$$

$$+$$

$$\begin{bmatrix} C_3 & -C_3 & 0 & 0 \\ -C_3 & C_2 + C_3 & -C_2 & 0 \\ & -C_2 & C_1 + C_2 & 0 \\ 0 & 0 & 0 & C_r \end{bmatrix} \begin{bmatrix} \dot{X}_3 \\ \dot{X}_2 \\ \dot{X}_1 \\ \dot{\theta} \end{bmatrix}$$

$$+$$

$$\begin{bmatrix} K_3 & -K_3 & 0 & 0 \\ -K_3 & K_2 + K_3 & -K_2 & 0 \\ 0 & -K_2 & K_1 + K_2 & 0 \\ 0 & 0 & 0 & K_r \end{bmatrix} \begin{bmatrix} X_3 \\ X_2 \\ X_1 \\ \theta \end{bmatrix} = \begin{bmatrix} -M_3\ddot{X}_0 \\ -M_2\ddot{X}_0 \\ -M_1\ddot{X}_0 \\ -(M_3H_3+M_2H_2)\ddot{X}_0 \end{bmatrix}$$

If there is no base rotation θ is zero and the matrix equation reduces to:

$$\begin{bmatrix} M_3 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_1 \end{bmatrix} \begin{bmatrix} \ddot{X}_3^{\,1} \\ \ddot{X}_2^{\,1} \\ \ddot{X}_1 \end{bmatrix} + \begin{bmatrix} C_3 & -C_3 & 0 \\ -C_3 & C_2 + C_3 & -C_2 \\ 0 & -C_2 & C_1 + C_2 \end{bmatrix} \begin{bmatrix} \dot{X}_3^{\,1} \\ \dot{X}_2^{\,1} \\ \dot{X}_1 \end{bmatrix}$$

$$+ \begin{bmatrix} K_3 & -K_3 & 0 \\ -K_3 & K_2 + K_3 & -K_2 \\ 0 & -K_2 & K_1 + K_2 \end{bmatrix} \begin{bmatrix} X_3^{\,1} \\ X_2^{\,1} \\ X_1 \end{bmatrix} = \begin{bmatrix} -M_3\ddot{X}_0 \\ -M_2\ddot{X}_0 \\ -M_1\ddot{X}_0 \end{bmatrix} \qquad (5.17)$$

If there is no base translation, $X_1 = 0$ and the matrix equation reduces to:

$$\begin{bmatrix} M_3 & 0 & M_3H_3 \\ 0 & M_2 & M_2H_2 \\ M_3H_3 & M_2H_2 & I_0 + M_3H_3^{\,2} + M_2H_2^{\,2} \end{bmatrix} \begin{bmatrix} \ddot{X}_3^{\,1} \\ \ddot{X}_2^{\,1} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} C_3 & -C_3 & 0 \\ -C_3 & C_2 + C_3 & 0 \\ 0 & 0 & C_r \end{bmatrix} \qquad (5.18)$$

$$\begin{bmatrix} \dot{X}_3^{\,1} \\ \dot{X}_2^{\,1} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} K_3 & -K_3 & 0 \\ -K_3 & K_2 + K_3 & 0 \\ 0 & 0 & K_r \end{bmatrix} \begin{bmatrix} X_3^{\,1} \\ X_2^{\,1} \\ \theta \end{bmatrix} = \begin{bmatrix} -M_3\ddot{X}_0 \\ -M_2\ddot{X}_0 \\ -(M_3H_3+M_2H_2)\ddot{X}_0 \end{bmatrix}$$

If there is no translation or rotation, $X_1 = 0$ and $\theta = 0$. The matrix equation then reduces to:

$$\begin{bmatrix} M_3 & 0 \\ 0 & M_2 \end{bmatrix} \begin{bmatrix} \ddot{X}_3{}^1 \\ \ddot{X}_2{}^1 \end{bmatrix} + \begin{bmatrix} C_3 & -C_3 \\ -C_3 & C_2 + C_3 \end{bmatrix} + \begin{bmatrix} K_3 & -K_3 \\ -K_3 & K_2 + K_3 \end{bmatrix}$$

$$\begin{bmatrix} X_3{}^1 \\ X_2{}^1 \end{bmatrix} = \begin{bmatrix} -M_3\ddot{X}_0 \\ -M_2\ddot{X}_0 \end{bmatrix} \qquad (5.19)$$

C. Multimass Extension of Equations

The methods used in the previous two sections to derive the two and three mass systems of equations can be, of course, used in an analogous manner to accomodate any system of the same form but with an increased number of lumped masses. However, comparing the final forms of the two and three mass system equations (see euations 5.8 and 5.16) it becomes apparent that the generalized "N" free mass system will take the following form.

$$\begin{bmatrix} M_N & 0 \cdots\cdots & & & & M_N H_N \\ 0 & M_{N-1} & 0 \cdots\cdots & & & \vdots \\ \cdots 0 & & M_{N-2} & 0 & & \vdots \\ \cdots\cdots & & \cdots 0 & * & & M_2 H_2 \\ \cdots\cdots & & & \cdots & M_1 & 0 \\ M_N H_N \cdots & & \cdots M_2 H_2 & & 0 & I_0 + \sum_{i=2}^{N} M_i H_i{}^2 \end{bmatrix} \begin{bmatrix} \ddot{X}_N \\ \ddot{X}_{N-1} \\ \vdots \\ \vdots \\ \ddot{X}_1 \\ \ddot{\theta} \end{bmatrix} \qquad (5.20)$$

$$
+
$$

$$
\begin{bmatrix}
C_N & -C_N & 0 & \cdots & \cdots & 0 \\
-C_N & C_N + C_{N-1} & -C_{N-1} & & & \vdots \\
0 & * & * & & * & \vdots \\
\vdots & & & * & & \vdots \\
\vdots & & & & * & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & C_r
\end{bmatrix}
\begin{bmatrix}
\dot{X}_N \\
\dot{X}_{N-1} \\
\vdots \\
\vdots \\
\dot{X}_1 \\
\dot{\theta}
\end{bmatrix}
$$

$$
+
$$

$$
\begin{bmatrix}
K_N & -K_N & 0 & \cdots & \cdots & \cdots & 0 \\
-K_N & K_N + K_{N-1} & -K_{N-1} & \cdots & \cdots & \cdots & \vdots \\
0 & * & * & & & & \vdots \\
\vdots & \vdots & & * & & & \vdots \\
\vdots & \vdots & & & * & & \vdots \\
\vdots & \vdots & & & & * & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & K_r
\end{bmatrix}
\begin{bmatrix}
X_N \\
X_{N-1} \\
\vdots \\
\vdots \\
X_1 \\
\theta
\end{bmatrix}
$$

$$
= \begin{bmatrix}
-M_N \ddot{X}_0 \\
-M_{1-1} \ddot{X}_0 \\
\vdots \\
\vdots \\
-M_1 \ddot{X}_0 \\
-\sum_{i=2}^{N} M_i H_i \quad \ddot{X}_0
\end{bmatrix}
$$

or, using a shortened notation

$$
M \ddot{X} + C \dot{X} + K X = F \tag{5.21}
$$

where M denotes the previous mass matrix

C denotes the previous damping coefficient matrix

K denotes the previous stiffness coefficient matrix

F denotes the inertial matrix.

VI. SELECTION OF PARAMETER VALUES

A. General

Any study of structure-foundation vibration requires at the outset a selection of appropriate structure and foundation mass, stiffness and damping values. The proper selection of these values is a subject which still requires much investigation. It is not the purpose of this section to offer any new information on this field, however. Rather, the literature on this subject is briefly and selectively reviewed with the intent of establishing the basis for the general range of values selected for a more detailed parametric examination.

Selection of meaningful structural stiffness values is on a firm calculational basis for normal structural elements. When structural elements have proportions that make questionable the application of a mechanics of materials approach or when complex structural systems are considered, some degree of uncertainity still exists, however. The selection of structural damping values, conversely, has little experimental or theoretical basis and much research in this area is still required.

The problem of characterizing the dynamic properties of a foundation is thought reducible to that of selecting appropriate values for three factors; effective foundation mass and inertia, foundation stiffness, and foundation percent of critical damping. It has been shown through research that the effective foundation mass is, to a good approximation, the mass of the foundation slab for structures of the type considered in this investigation. For foundation stiffness, recent advances in the theory of vibration of rigid bodies on elastic foundations and new data from large field tests now enable the analyst to make at least a good order of magnitude determination of foundation stiffness values. The state-of-the-art with respect to selection of suitable foundation damping values is not as far advanced. It parallels roughly the structural damping situation.

B. Structural Stiffness and Mass

The stiffness of individual structural elements is commonly known and can be calculated precisely. When considering an assemblage of such elements, as indicated previously, the calculation can be quite complex and can, for many situations, be made in only an approximate manner. Blume, Newmark and Corning (6) discuss this problem extensively. For the structures of this investigation the structural framing system can, with sufficient accuracy, be considered as single or multiple shear walls. For such shapes the deflection due to lateral load can be represented by the formulas:

For shear: $\frac{FL}{AG}$ (6.1)

For flexure: $\frac{FL^3}{\cancel{2}EI}$ (6.2)

The combined stiffness then becomes

$$K = \frac{F}{\Delta F + \Delta S}$$

$$= \frac{F}{\frac{FL}{AG} + \frac{FL^3}{EI}}$$

$$= \frac{EIAG}{L(AGL^2 + k EI)} \qquad (6.3)$$

The procedures used and the results of this study are not dependent upon a precise determination of stiffness inasmuch as all results are expressible in terms of the relation between foundation and structural stiffness. However, a better than order of magnitude determination is desirable to establish the approximate upper and lower limits of containment structural stiffness and, also, to establish the foundation to structure stiffness ratios of interest. The value of k must range between 12.0 (corresponding to fixed pier action) and 3.0 (corresponding to cantilever pier action). If the containments were open at the top (side walls only) the 3.0 value would be applicable. If the containment dome (and ring beam for the prestressed concrete containments) sections are very stiff and massive with respect to the walls of the structure the 12.0 value would be applicable. The actual situation lies somewhat between these two situations. For this study a value of 12.0 was arbitarily selected.

In addition, the following values of E (modulus of elasticity) and G (shear modulus) were selected as appropriate for concrete:

$E = 3 \times 10^6$ psi (Type III, IV and V structures)

$= 5 \times 10^6$ psi (Type I and II structures)

$G = 1/3$ of corresponding E value

The lumping of masses is a feature of the structural model for which equations of motion were developed in the previous section. It is desirable to assign values to these masses (M_1 and M_2) such that the actual non-uniformly distributed mass of the structure is accurately dynamically modeled. The guidance that is available in this area is generally related to the dynamically modeling of a distributed mass with an equivalent mass in a single degree of freedom system. Inasmuch as the system considered in this study is not reducible to a single degree of freedom except in special cases, this technique is not applicable.

For this research three criteria for mass lumping were used. For all stiffness and damping investigations the total moment of inertia of the containment structure was known with acceptable accuracy. The location of top mass (M_2) concentration was selected as the center of mass of the containment dome and the containment base mass (M_1) was positioned at the center of the containment base since these were the two points of extraordinary mass concentration. The value of H_2 was fixed as the distance between these two masses. The top mass (M_2) was then sized by requiring $M_2H_2{}^2$ to equate to the total containment structure moment of inertia about the center of mass of the base. M_2H_2, in turn, equals this total moment divided by H_2.

The mass M_1 value was selected such that the sum of the two masses (M_1 and M_2) totaled to the containment mass. This criteria is called the "I^1 equated top mass" criterion.

An alternate mass sizing criterion, called the "Half wall top mass" criteria was used for comparative purposes in part of the stiffness investigation. The values of total containment mass moment and mass moment of inertia were, of course, known values. The mass moment of inertia of the containment was preserved and the M_2H_2 value was specified so as to preserve total containment mass moment. The masses M_1 and M_2 were determined by conceptually slicing the containment at mid height. All mass above this slice was assigned to the top mass and all mass below this slice was assigned to the bottom mass.

To obtain an idea of the extent to which mass sizing could influence structural response, two other mass selections were used in a limited number of cases. The previously described procedure was used to specify mass moment and mass moment of inertia. To establish M_1 and M_2 the conceptual slicing of the second procedure was used with mass partitions corresponding to slices at both the top and bottom third points being alternately considered. These cases are identified in the figures as the "One-third (two-thirds) side wall mass" criteria.

C. Structural Damping

Damping in structures has been investigated in only a very limited manner. Much of the research that has been done has, furthermore, been related to the damping characteristic of samples. A summary of this work has been presented by Cole (9). Logarithmic energy decrements in the range of 0.02 to 0.03 have generally been reported for uncracked specimens. Where cracking occurred, the decrement was increased. No appreciable amplitude or frequency dependence has been observed.

Some information has been compiled by investigators on the damping characteristics of complete buildings. The results of twenty tests on reinforced concrete shear type buildings has, as an example, been tabulated in table one. In these tests the buildings listed were all excited by mounting a vibrator in each structure and measuring the response at selected locations. Investigators at the test sites generally commented that they felt that greater damping could be anticipated in earthquakes of damaging intensity. The values listed in table one are reported first mode values. The difficulties involved in exciting higher modes have restricted the amount of data available on the damping of the higher modes. The limited data available, however, supported a premise that damping is independent of mode (8).

At the present time, the type of damping that is most characteristic of structural systems has also not been established. However, viscous damping is, perhaps, easier to handle mathematically than other possible types. It has, morever, been shown by Jacobsen (19) that the assumption of viscous damping is a justifiable approximation to other types of damping for forced vibration problems.

For this investigation viscous damping has been used. In use of this damping, coefficients have been selected by the procedure of Lycan and Newmark (24). Specifically the structural damping coefficient C_2 was evaluated by considering the structure as fixed base. The coefficient C_2 then is related by single degree-of-freedom theory to the mass of the structure, stiffness of the structure, and the known (or assumed) percentage of critical damping in the structure by the relationship $C_2 = 2f\sqrt{MK}$. Application of this same procedure yields C_r and C_1 in an analogous manner. The value chosen for f in the damping phase of this investigation is 0.02. The fact that this represents a conservative but realistic value for damping in structural concrete can be seen by referral to table one.

D. Foundation Stiffness

With the analytical approach being used in this research it is required that foundation stiffnesses for foundation translation and rocking be known. Probably the best information in this area is available through the research of Barkan (1). Barkan has shown these coefficients to be a function of soil type, size of foundation, and, to a limited degree, the geometry of the foundation. He has tabulated approximate ranges of values for these constants for broad soil classes. Where it is required for a specific site that these constants be determined fairly precisely, small scale field testing or accurate determination of soil elastic constants is required. However, the general range of values listed by Barkan is more useful for this investigation. Barkan's data shows the coefficient of

Table 1. Damping in reinforced concrete buildings by vibrator testing (29)(5)(22)

Structure	Investigator	Number of stress	C/C_r
Kumegawa I	Nahagawa	5	0.045
Kumegawa II	"	4	0.036N
			0.069E
Tahemaru I	"	4	0.093
Tahemaru II	"	4	0.057
Nishinomuja	"	4	0.115
Komagoma	"	4	0.037
Senjyu	"	4	0.069
Totsuha	"	4	0.048
Honmuracho	"	4	0.095
Ishihawacho	"	4	0.050
Gotohyji	"	4	0.038
Taishido	"	4	0.036
Kyodo	"	4	0.071
Number Five	Meehan and Blume	1	0.047
Number Seven	"	2	0.08N
			0.07E
Number Eight	"	2	0.124
Number Nine	"	2	0.077N
			0.094E
Number Ten	"	3	0.079N
			0.076E
Aoyama	Kanai and Yoshizawa	3	0.042N
			0.110E
Tahaba	"	4	0.070

elastic vertical stiffness (defined as the vertical force per unit area in a soil mass to cause unit vertical deflection) to range from three to ten Kg/cm^3. He further suggests that the corresponding coefficient of rocking stiffness can be taken as twice the vertical stiffness value. The coefficient of translational stiffness is suggested to be half the vertical value. The range selected for this investigation, based on the information outlined above, is from one Kg/cm^3 (considered as a very soft foundation) to ten Kg/cm^3 (considered as a stiff foundation). The range is further extended to 100 Kg/cm^3 (to represent pile systems) and beyond to represent conditions approaching good, solid rock. Using these unit properties integrated stiffnesses are obtained by multiplying by the foundation contact area for the translational value and by multiplying by the second moment of the contact area about the horizontal axis through the foundation centroid and normal to the plane of rocking for the rotational value.

E. Foundation Damping

Foundation damping comes from two sources, by radiation of energy away from the structure in the soil wave induced in the soil by structure motion and by inelastic behavior of the soil mass itself. Whitman (43) states that the radiative contribution is greater for horizontal translation than it is for rocking motion. The inelastic soil action contribution does not appear dependent on type of motion but is known to be a function of soil type and ground moisture conditions. Little quantitative information is available.

Barkan (1) tabulates experimentally determined damping values and discusses the effect of partial foundation embedment. His damping values for exposed foundations range from five to twenty percent. Where partial embedment is featured Barkan indicates that damping values as high as three and one-half times this amount have been obtained.

Sufficient information does not appear yet available to permit evaluation of the damping developable by any specific foundation design. Presently available information does indicate that foundations do have the capability to develop large amounts of damping, however, and because of this fact the foundation damping in this study was varied through a range from two to forty percent. Information of a quantitative nature will, of course, eventually be available and may even enable some control by the foundation designer of foundation damping properties.

Conversion of percentages of critical damping to usable damping coefficients for this investigation, as with the structural damping, follows the technique of Lycan and Newmark (24). The rotational damping coefficient is selected to be proportional to the circular frequency of the foundation rigid structure translational mode.

VII. ANALOG COMPUTER ANALYSIS

A. Forcing Function Development

In any investigation of structural response to earthquake ground motion it is necessary to specify precisely what constitues earthquake ground motion. This problem has no complete answer at present and is, in fact, the current topic of much research by earthquake investigators. Two currently considered general approaches to the problem of simulating earthquake motion are the probabilistic approach and the deterministic approach. In the probabilistic approach the earthquake is treated as a random process. That an earthquake is inherently somewhat random in characteristic is obvious when one notes the variability of earth stratum an earthquake shock wave will move through on its passage to any specific surface location. However, the specific type of random process most suitable for earthquake representation is still widely debated. Bycroft (7), by way of example, proposes the use of a "white noise" having a constant spectral density of 0.75 $ft^2/sec^4/cps$ with a duration of thirty seconds.

More successful from the standpoint of past application, however, has been the deterministic approach. In this approach actual or simulated earthquake records are used as forcing functions in conjunction with a modal or numerical structural technique.

Both the probabilistic and deterministic approaches have their merits and demerits. It was the choice for this investigation to use the deterministic approach due to its extensive current use in actual practice. Having once decided to use this approach criteria were then needed for selection of a specific forcing function and to gage its acceptability. The only criteria decided upon were a) that the response spectra from the forcing function would reasonably simulate in general shape and magnitude the averaged response spectra of a typical earthquake (El Centro) and b) that the response spectrum from the forcing function would be of smooth shape in order to avoid the difficulties that a irregular shape of spectrum would present in the analysis. The development of a suitable function then proceeded in a "cut and try" manner until a function meeting the above criteria was achieved. The analog setup used to generate the half-cycle sine function finally decided upon is shown in figures nine and ten.

The response spectrum for the chosen function (or any other) can be developed from the following mathematical considerations. Consider a single free mass, one degree-of-freedom oscillator. The equation of motion of this system for ground motion is

$$M_1\ddot{X}_1 + C_1(\dot{X}_1 - \dot{X}_0) + K_1(X_1 - X_0) = 0 \qquad (7.1)$$

or

$$M_1\ddot{X}_1 + C_1\dot{X}_1 + K_1X_1 - C_1\dot{X}_0 - K_1X_0 = 0 \qquad (7.2)$$

Making the substitution $u = X_1 - X_0$ and, likewise, $\dot{u} = \dot{X}_1 - \dot{X}_0$ and $\ddot{u} = \ddot{X}_1 - \ddot{X}_0$ yields

$$M_1\ddot{u} + C_1\dot{u} + K_1u = -\ddot{X}_0M_1 \qquad (7.3)$$

Rearranging and substituting $\omega^2 = \frac{K_1}{M_1}$ and $2\,\omega f = \frac{C_1}{M_1}$ yields

$$\ddot{u} + 2\,\omega f\dot{u} + \omega^2u = -\ddot{X}_0 \qquad (7.4)$$

From the preceding development it is observed that the ground displacement has been transformed into an equivalent second order ordinary differential equation with the ground acceleration as a forcing function. Now considering again the equation of motion of the system, it is to be observed that it can also be written in the following equivalent form:

$$M\ddot{X}_1 + C_1\dot{u} + K_1u = 0 \qquad (7.5)$$

Noting that at the relative maximum displacement $\dot{u} = 0$, the equation at u_{max} then reduces to

$$M\ddot{X}_1 + K_1u_m = 0 \qquad (7.6)$$

or

$$\ddot{X}_1 = -\,\omega^2u_m \qquad (7.7)$$

The value of $\ddot{X}_1$ associated with u_m is designated the spectral acceleration. The spectral velocity associated with this apparent harmonic motion can likewise be obtained by noting that for simple harmonic motion

$$\dot{u}_m = \omega u_m \qquad (7.8)$$

From the above considerations, a tabulation of maximum relative displacements when the single degree-of-freedom system is subjected to an accelogram forcing function is seen to yield spectral acceleration, velocity and displacement. The analog setup used to obtain these parameters is shown in figure eleven. The spectral displacement for this function and, for comparison, a generalized El Centro spectrum are shown in figures twelve and thirteen respectively. The El Centro and the selected forcing function spectra do not coincide completely except in general shape and frequency range. However, there is no necessity that they do coincide and, in fact, comparison of any two earthquake spectra would be expected to show a similar variation. It can be observed that the chosen acceleration function does, in fact, meet the criteria previously set forth, however.

B. Analog Equation Formulation

In section V the general two free mass equations are presented in the following form:

$$M_2\ddot{X}_2 + M_2H_2\ddot{\theta} + C_2\dot{X}_2 - C_2\dot{X}_1 + K_2X_2 - K_2X_1 = -M_2\ddot{X}_0 \qquad (7.9)$$

$$M_1\ddot{X}_1 + C_1\dot{X}_1 - C_2\dot{X}_2 + C_2\dot{X}_1 + K_1X_1 - K_2X_2 + K_2X_1 = -M_1\ddot{X}_0$$

$$M_2H_2\ddot{X}_2 + M_2H_2^2\ddot{\theta} + I_0\ddot{\theta} + C_r\dot{\theta} + K_r\theta = -M_2H_2\ddot{X}_0$$

Rearranging these equations so that the highest ordered derivative is alone on the opposite side of the equation and dividing through by its coefficient yields the following equations:

$$\ddot{X}_2 = -\frac{C_2}{M_2}(\dot{X}_2 - \dot{X}_1) - \frac{K_2}{M_2}(X_2 - X_1) - H_2\ddot{\theta} - \ddot{X}_0 \qquad (7.10)$$

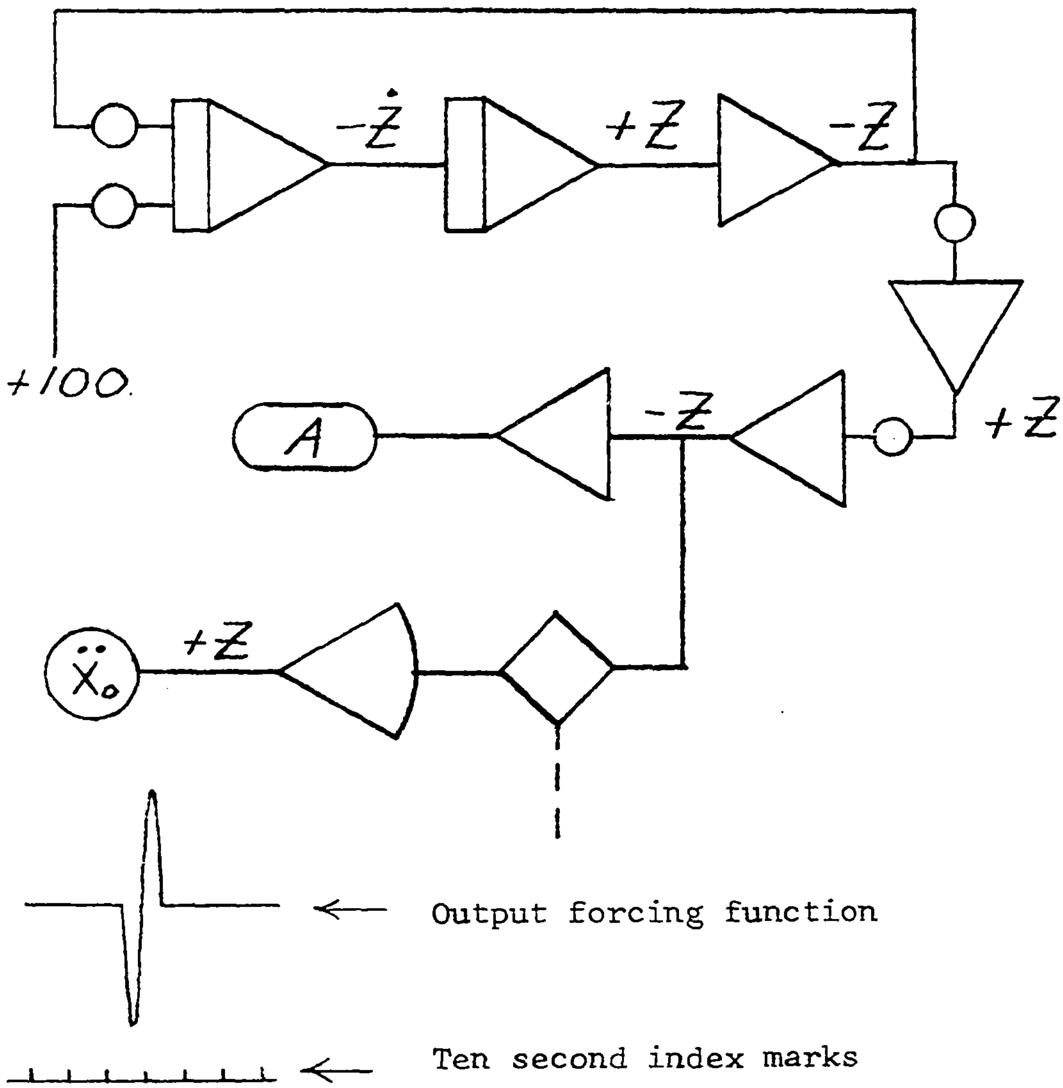


Figure 9. Analog schematic for function generator

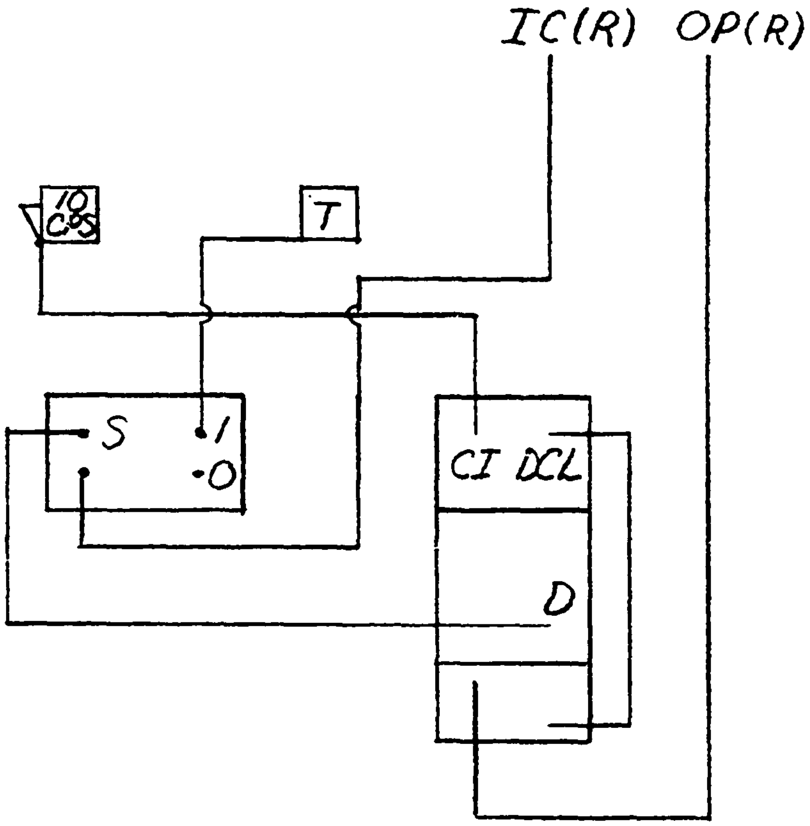


Figure 10. Analog schematic for function generator switching logic

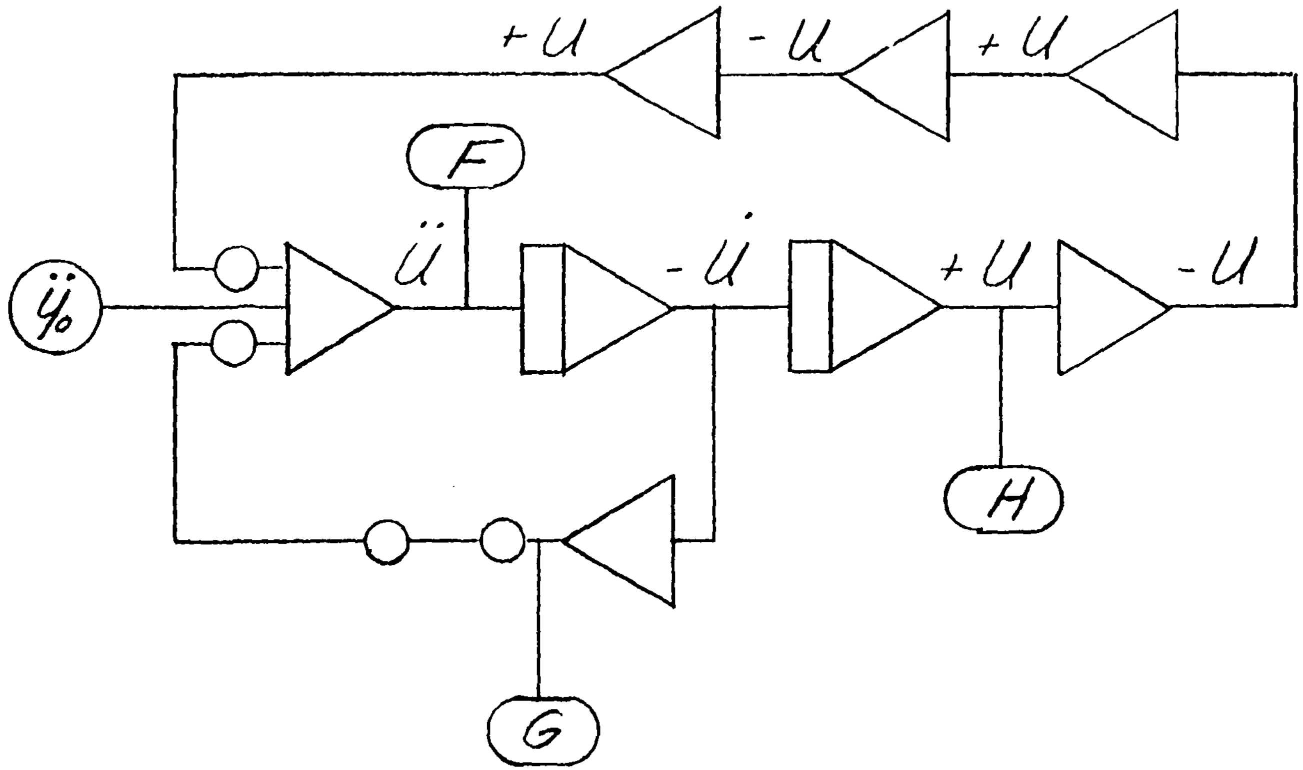


Figure 11. Analog schematic for response spectrum determination

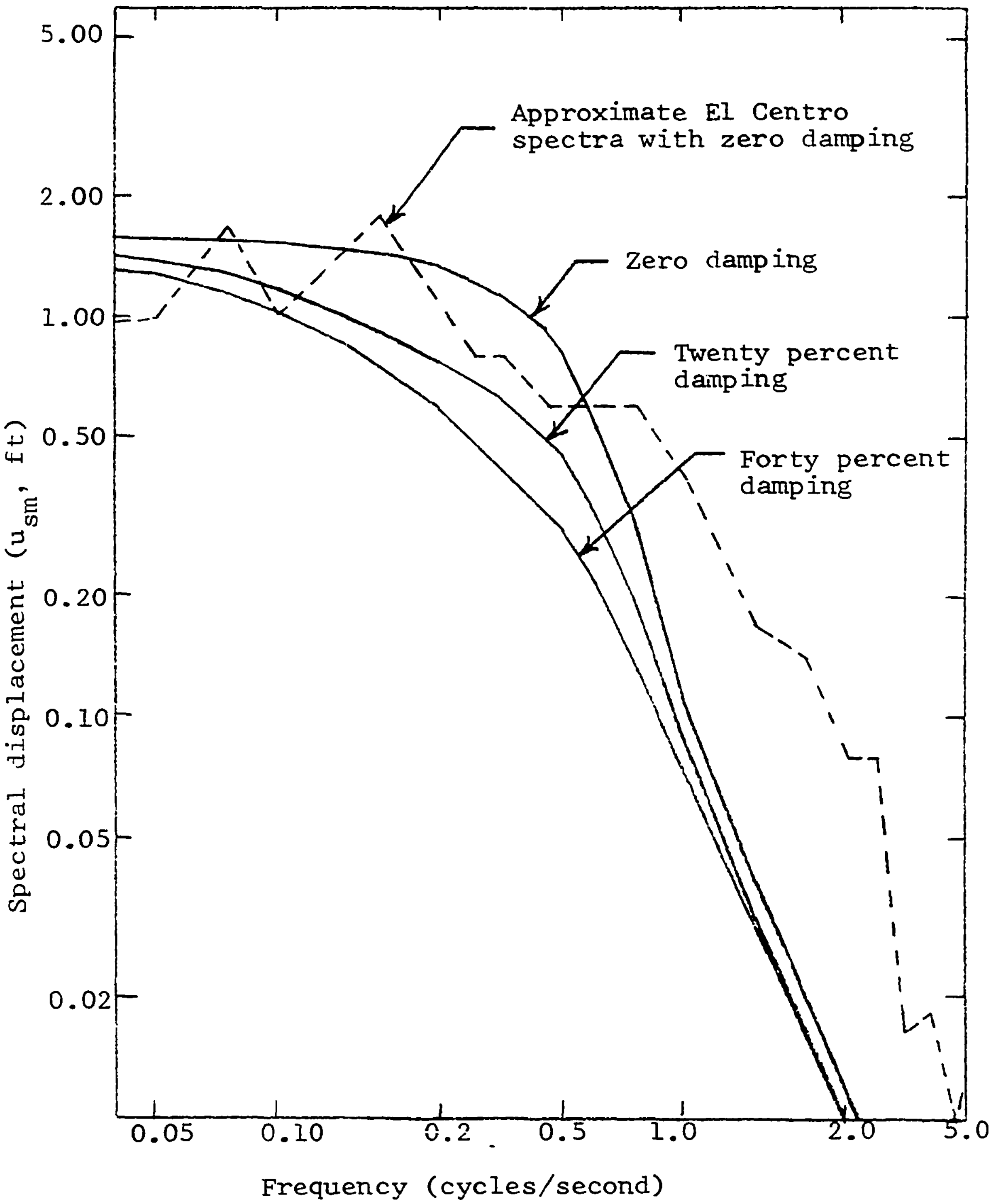


Figure 12. Displacement response spectrum, simulated earthquake ground motion

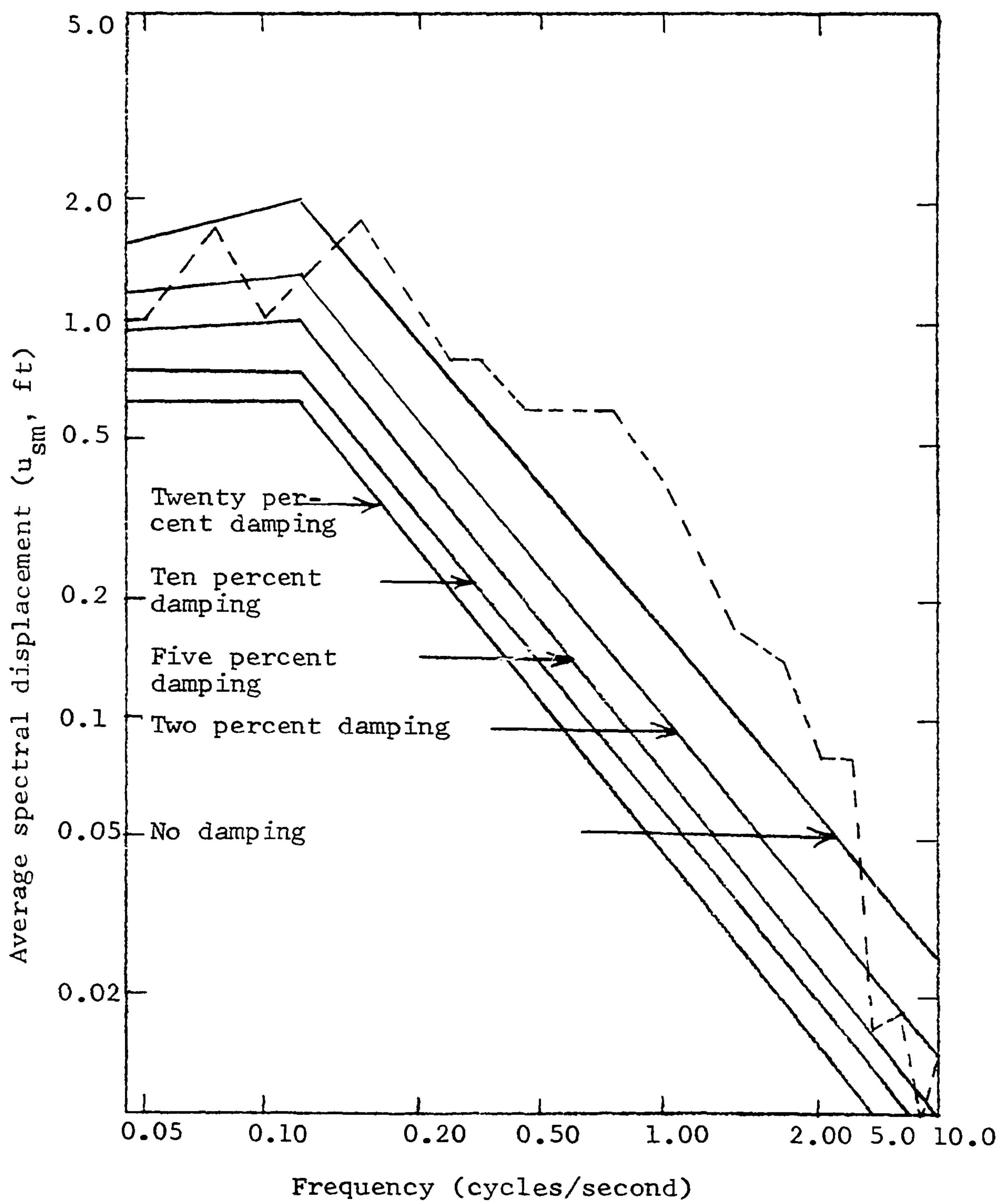


Figure 13. Average displacement response spectrum, El Centro earthquake, May 18, 1940 (N-S direction)

$$\ddot{X}_1 = -\frac{C_1}{M_1}\dot{X}_1 - \frac{K_1}{M_1}X_1 + \frac{C_2}{M_1}(\dot{X}_2 - \dot{X}_1) + \frac{K_2}{M_1}(X_2 - X_1) - \ddot{X}_0$$

$$\ddot{\theta} = -\frac{C_r}{M_2H_2^2 + I_0}\dot{\theta} - \frac{K_r}{M_2H_2^2 + I_0}\theta - \frac{M_2H_2}{M_2H_2^2 + I_0}\ddot{X}_2$$

$$-\frac{M_2H_2}{M_2H_2^2 + I_0}\ddot{X}_0$$

Inputing $\ddot{X}_0$ as a forcing function into the analog computer, the $\dot{\theta}$, $\dot{X}_1$, θ, X_1 and X_2 responses can be observed, all in terms of system voltages. To make the solution compatible with the EAI 8800 analog computer magnitude and time scaling are required as a preliminary step, however. This can be accomplished using the relationships

$X_1 = \beta y_1$, $X_2 = \beta y_2$, $\theta = \gamma\theta^1$, and $\mathcal{T} = \alpha t$ where y is an analog voltage, $\mathcal{T}$ is the analog time, and β, γ, $\mathcal{T}$ represent scaling constants. Differentiating these expressions with respect to time yields

$$\frac{dX}{dt} = \alpha\beta\frac{dy}{d} \qquad \frac{d^2X}{dt^2} = \alpha^2\beta\,\frac{d^2y}{d\mathcal{T}^2}$$

Now substituting these relationships into the previous two mass system equations yields

$$\beta\ddot{y}_2 = -\frac{C_2}{M_2\alpha}[(\beta\dot{y}_2) - (\beta\dot{y}_1)] - \frac{K_2}{M_2\alpha^2}[(\beta y_2) - (\beta y_1)]$$

$$- H_2(\gamma\ddot{\theta}^1) - \frac{(\beta\ddot{y}_0)}{\alpha^2} \qquad (7.11)$$

$$\beta \ddot{y}_1 = - \frac{C_1}{M_1}(\beta \dot{y}_1) - \frac{K_2}{M_1\alpha^2}(\beta y_1) + \frac{C_2}{M_1\alpha}[(\beta \dot{y}_2) - (\beta \dot{y}_1)] + \frac{K_2}{M_1\alpha^2}$$

$$[(\beta y_2) - (\beta y_1)] - \frac{(\beta \ddot{y}_0)}{\alpha^2}$$

$$\gamma \ddot{\theta}^1 = - \frac{C_r}{(M_2H_2^2 + I_0)\alpha}(\gamma \dot{\theta}^1) - \frac{K_r}{(M_2H_2^2 + I_0)\alpha^2}(\gamma \theta^1) - \frac{M_2H_2}{(M_2H_2^2 + I_0)}$$

$$(\beta \ddot{y}_2) - \frac{M_2H_2}{(M_2H_2^2 + I_0)\alpha^2} (\beta \ddot{y}_0)$$

Considering machine capabilities and the particular range of magnitude variables and the time variation associated with the type of problem being solved, it is desirable to make $\alpha = 10$, $\beta = 0.10$ and $\gamma = 0.001$. Substituting these values into the equations and dividing through by β and γ yields:

$$\ddot{y}_2 = - \frac{C_2}{10M_2} (\dot{y}_2 - \dot{y}_1) - \frac{K_2}{100M_2} (y_2 - y_1) - \frac{H_2\ddot{\theta}^1}{10} - \frac{\ddot{y}_0}{100}$$

$$\ddot{y}_1 = \frac{C_1 y_1^1}{10M_1} - \frac{K_1y_1}{100M_1} + \frac{C_2}{10M_1} (\dot{y}_2 - \dot{y}_1) + \frac{K_2}{100M_1} (y_2 - y_1) - \frac{\ddot{y}_0}{100} \tag{7.12}$$

$$\ddot{\theta} = - \frac{C_r\dot{\theta}^1}{10(M_2H_2^2 + I_0)} - \frac{K_r\theta^1}{100(M_2H_2^2 + I_0)} - \frac{100M_2H_2\ddot{y}_2}{M_2H_2^2 + I_0}$$

$$- \frac{M_2H_2\ddot{y}_0}{M_2H_2^2 + I_0}$$

Considering the controlling values of the data from Appendix B yields the following set of equations with bounding values:

$$\ddot{y}_2 = -\begin{pmatrix}0.1090\\0.5890\end{pmatrix}(\dot{y}_2 - y_1) - \begin{pmatrix}7.34\\34.80\end{pmatrix}(y_2 - y_1) - \begin{pmatrix}1.300\\2.500\end{pmatrix}\ddot{\theta} - \frac{\ddot{y}_0}{100}$$

$$\ddot{y}_1 = -\begin{pmatrix}0.0325\\2.1020\end{pmatrix}\dot{y}_1 - \begin{pmatrix}0.1500\\17.5000\end{pmatrix}y_1 + \begin{pmatrix}0.0240\\0.4260\end{pmatrix}(\dot{y}_2 - \dot{y}_1) + \begin{pmatrix}1.6200\\0.4260\end{pmatrix}(\dot{y}_2 - \dot{y}_1) + \frac{\ddot{y}_0}{100} \tag{7.13}$$

$$\ddot{\theta} = -\begin{pmatrix}0.0234\\3.0280\end{pmatrix}\ddot{\theta}^{1} - \begin{pmatrix}0.4000\\14.3000\end{pmatrix}\theta^{1} - \begin{pmatrix}0.3000\\0.7100\end{pmatrix}\ddot{y}_2 - \begin{pmatrix}0.3000\\0.7100\end{pmatrix}\frac{\ddot{y}_0}{100}$$

The analog schematics for the setup of this set of equations are shown in figures fourteen, fifteen and sixteen.

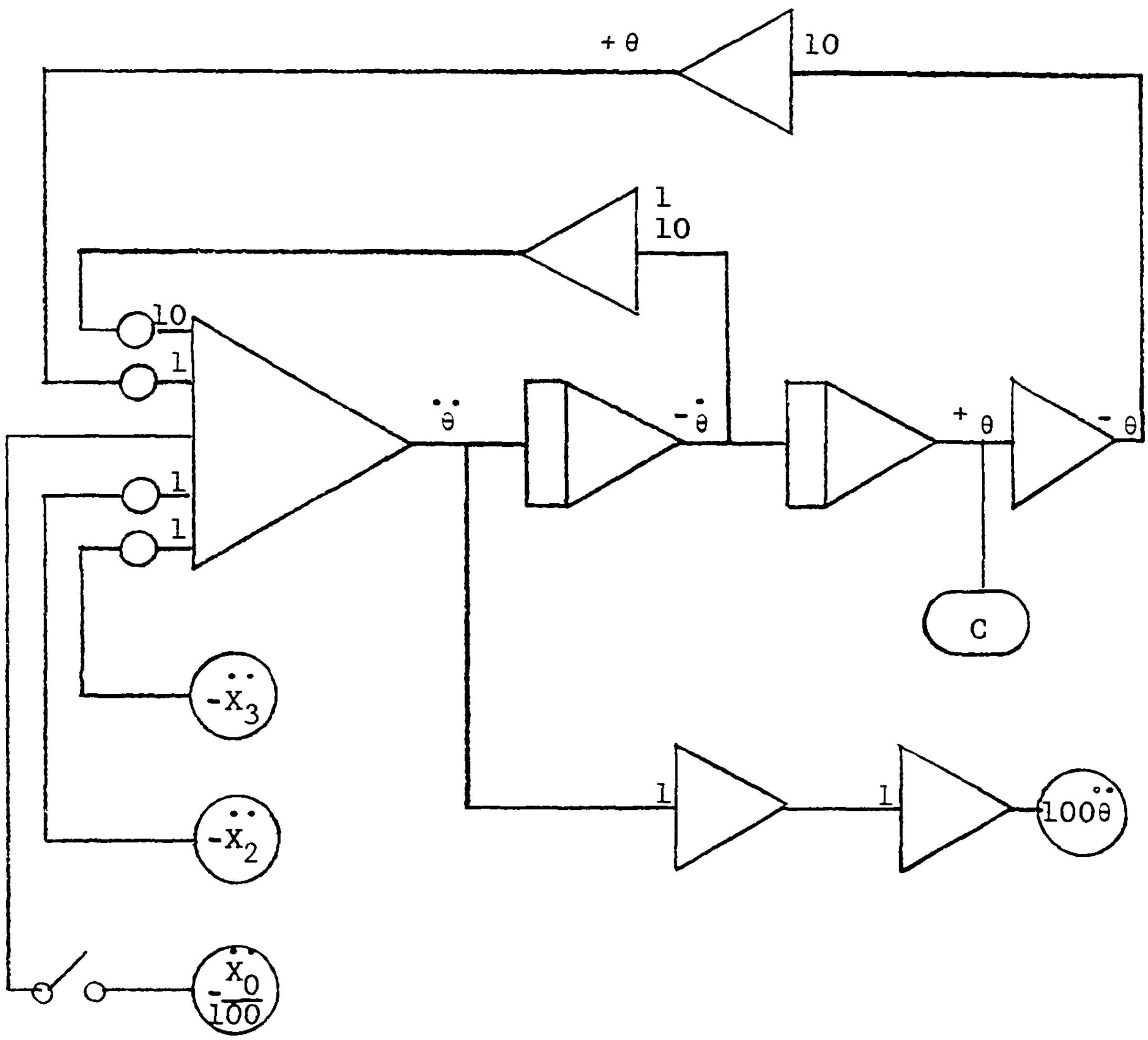


Figure 14. Analog schematic for rocking motion

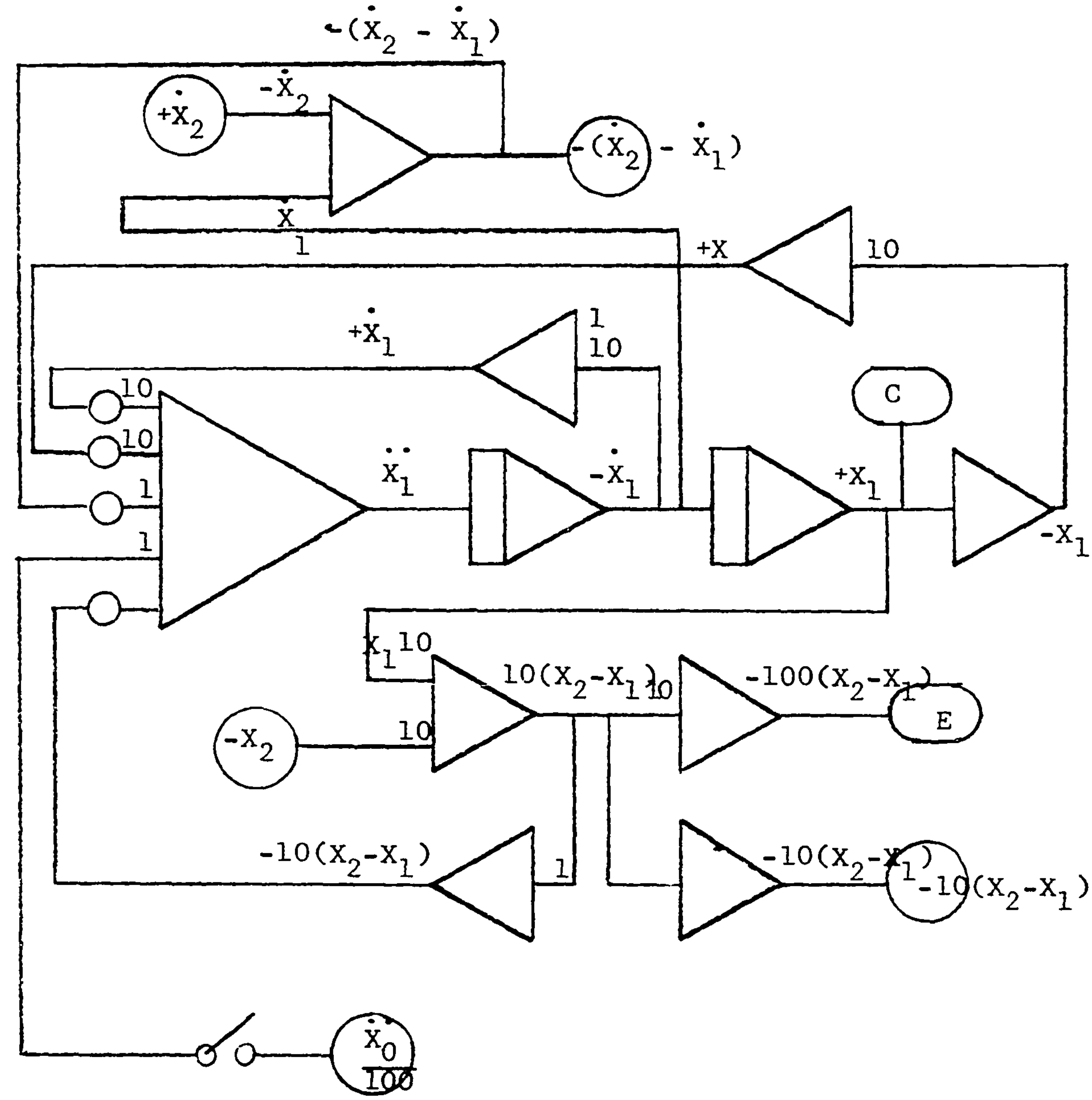


Figure 15. Analog schematic for first mass motion

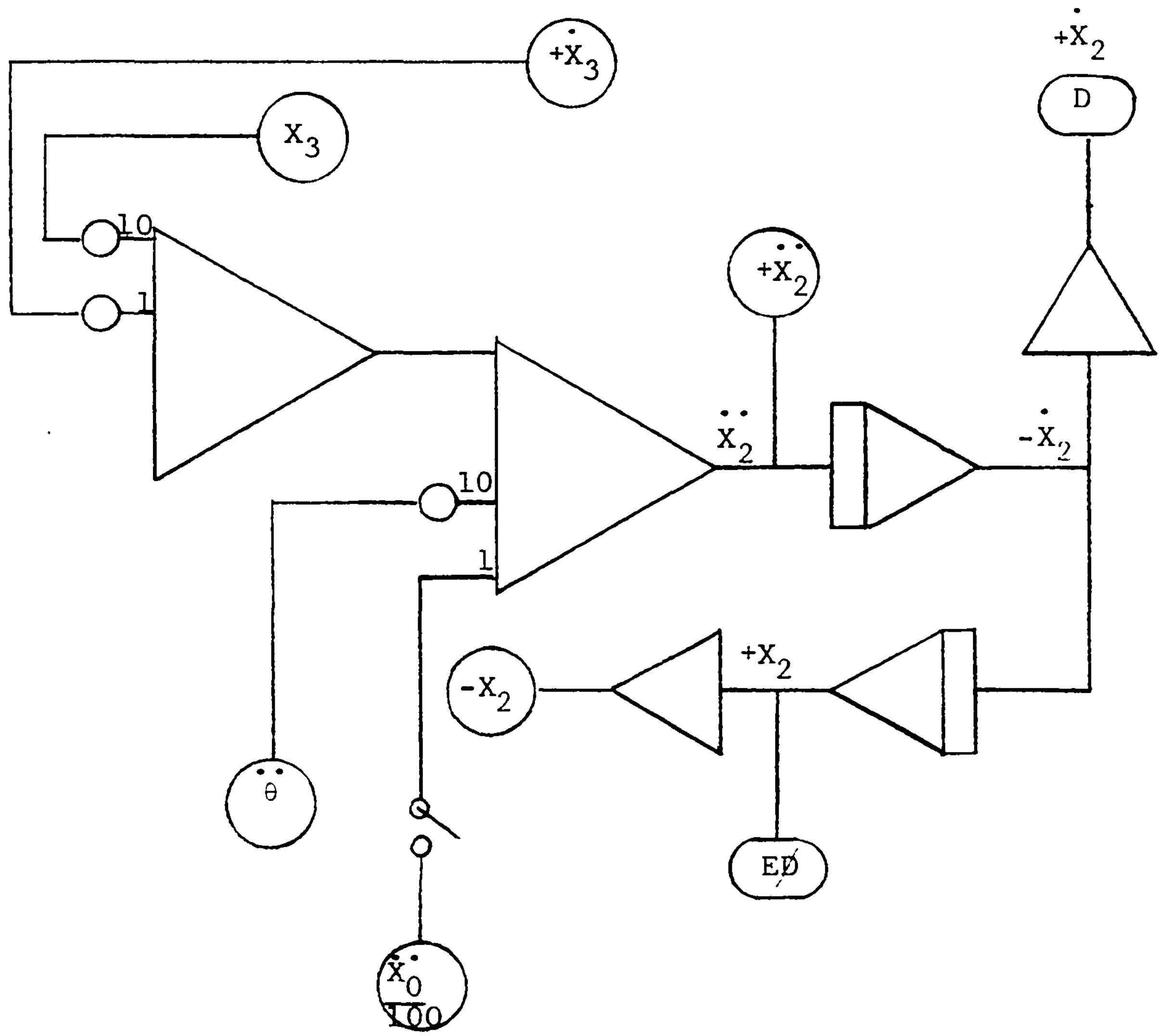


Figure 16. Analog schematic for second mass motion

VIII. GENERAL MATHEMATICAL ANALYSIS

A. Review of Methods

The method of solution most commonly used and perhaps presently best adapted to the solution of multidegree of freedom problems is the modal analysis method. The principle underlying the usefulness of this method is the fact that the differential equations of motion are decoupled when the displacements of the system are expressed in terms of natural modes of free undamped system vibration. Consider the equations of motion of a undamped multimass system with rocking. From section V equation (5.20), these equations become:

$$
\begin{aligned}
K_N X_N - K_N X_{N-1} &= -M_N \ddot{X}_N - M_N H_N \ddot{\theta} \\
-K_N X_N + (K_N + K_{N-1}) X_{N-1} - K_{N-1} X_{N-2} &= -M_{N-1} \ddot{X}_{N-1} - M_{N-1} H_{N-1} \ddot{\theta} \\
\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots \\
-K_2 X_2 + (K_1 + K_2) X_1 &= -M_1 \ddot{X}_1 \\
K_r \theta &= -\sum_{i=2}^{N} M_i H_i^2 \ddot{\theta} \\
&\quad -\sum_{i=2}^{N} M_i H_i \ddot{X}_i \\
&\quad - I_0 \ddot{\theta}
\end{aligned}
\tag{8.1}
$$

Assuming a solution of the form

$$X_i = y_i e^{i\omega t} \quad \text{and} \quad \theta = \Phi e^{i\omega t}$$

exists, these identities are substituted into the above equations to yield

$$K_N y_N - K_N y_{N-1} = (M_N y_N + M_N H_N \Phi)\omega^2$$

$$-K_N y_N + (K_N + K_{N-1}) y_{N-1} - K_{N-1} y_{N-2} = (M_{N-1} y_{N-1} + M_{N-1} H_{N-1} \Phi)\omega^2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-K_2 y_2 + (K_1 + K_2) y_1 = (M_1 y_1)\omega^2 \tag{8.2}$$

$$K_r \Phi = +\sum_{i=2}^{N} M_i H_i^2 \Phi \omega^2 + \sum_{i=2}^{N} M_i H_i y_i \omega^2 + I_0 \Phi \omega^2$$

This set of equations can be expressed more compactly in matrix form as

$$KY = \omega^2 MY \tag{8.3}$$

where

$$Y = \{y_N, y_{N-1}, y_{N-2}, \cdots y_1, \theta\}$$

$$K = \begin{bmatrix} K_N & -K_N & & \cdots \\ -K_N & +K_N + K_{N-1} & & \cdots \\ \cdot & \cdot & * & \\ \cdot & \cdot & & * \\ \cdot & \cdot & & & K_r \end{bmatrix}$$

and

$$M = \begin{bmatrix} M_N & \cdots & & & & M_N H_N \\ \cdot & M_{N-1} & & & & \cdot \\ \cdot & & * & & & \cdot \\ \cdot & & & * & & \cdot \\ \cdot & & & & * & \cdot \\ M_N H_N & \cdots & & & & I_0 + \sum_{i=2}^{N} M_i H_i^2 \end{bmatrix}$$

Rearranging then yields:

$$[K - \omega^2 M]Y = 0 \tag{8.4}$$

This set of homogenous linear equations can only have a solution if the determinant of the coefficients vanishes. Upon expanding this equation an N+1 order equation in ω^2 is obtained. In turn, for each value of ω^2 there exists a particular set of values called Y_i. Denoting two such vectors Y_i and Y_j, their corresponding ω's as ω_i and ω_j. Using the initial matrix equation yields:

$$KY_i = \omega_i^2 MY_i \tag{8.5}$$

$$KY_j = \omega_j^2 MY_j \tag{8.6}$$

Transposing equation (8.5), postmultiply by Y_j and premultiplying equation (8.6) by Y_i^T yields:

$$Y_i^T K^T Y_j = \omega^2 Y_i^T M^T Y_j \tag{8.7}$$

$$Y_i^T KY_j = \omega^2 Y_i^T MY_j$$

Noting that M and K are symmetric matrices, $K^T = K$ and $M^T = M$. These equations then become:

$$Y_i^T KY_j = \omega_i^2 Y_i^T MY_j \tag{8.8}$$

$$Y_i^T KY_j = \omega_j^2 Y_i^T MY_j$$

Observing that the left hand sides of the two equations (8.8) are equal, the right hand sides must also be equal. Equating then yields:

$$(\omega_i^2 - \omega_j^2) Y_i^T MY_j = 0 \tag{8.9}$$

Since ω_i^2 and ω_j^2 are not, in general, equal it follows that:

$$Y_i^T MY_j = 0 \tag{8.10}$$

Restating, it is to be noted that the eigenvectors are orthogonal with respect to the weighting matrix M. If $\omega_i^2 = \omega_j^2$ and $Y_i = Y_j$, $\omega_i^2 - \omega_j^2 = 0$ and $Y_i^T MY_i \neq 0$. Defining

$L_i^2 \equiv Y_i^T M Y_i$ and dividing by L_i^2 yields:

$$\frac{Y_i^T M Y_i}{L_i^2} = 1 \tag{8.11}$$

or

$$\frac{Y_i^T}{L_i} M \frac{Y_i}{L_i} = 1 \tag{8.12}$$

Letting the quantity $\frac{Y_i}{L_i}$ be denoted as the unit vector e_i with elements $\left\{ \frac{y_{N,i}}{L_i}, \frac{y_{N-1,i}}{L_i}, \dots \frac{y_{1,i}}{L_i}, \frac{\theta_i}{L_i} \right\}$

the equation can be rewritten as

$$e_i^T M e_i = 1 \tag{8.13}$$

Vectors that satisfy this equation are convectionally referred to as normalized vectors or unit vectors. Combining the unit vectors columnwise into a matrix designated the Q matrix, defined as $[e_1, e_2, \dots e_{N+1}]$ where e_i is further defined as $\{e_{1,i}, e_{2,i}, \dots e_{N+1,i}\}$ with these elements corresponding to $\left\{ \frac{y_{N,i}}{L_i}, \frac{y_{N-1,i}}{L_i}, \dots, \frac{y_{1,i}}{L_i}, \frac{\theta}{L_i} \right\}$ All possible relations of the form (8.13) then can be expressed in the more compact form:

$$Q^T M Q = I \tag{8.14}$$

Returning now to equation (8.5) and dividing by L_i yields

$$\frac{KY_i}{L_i} = \omega_i^2 M \frac{Y_i}{L_i} \tag{8.15}$$

or

$$Ke_i = \omega_i^2 Me_i \tag{8.16}$$

Combining all N+1 e_i-vectors into a more compact form yields

$$KQ = MQ\Lambda \tag{8.17}$$

where

$$\Lambda = \begin{bmatrix} \omega_1^2 & & & & \\ & \omega_2^2 & & 0 & \\ & & \cdot & & \\ 0 & & & \cdot & \\ & & & & \cdot\ \omega_{N+1}^2 \end{bmatrix}$$

Premultiplying by Q^T yields

$$Q^TKQ = Q^TMQ\Lambda \tag{8.18-8.19}$$

Note that it was previously shown (8.14) that $Q^TMQ = I$. Substituting this result into (8.18) yields

$$Q^TKQ = \Lambda \tag{8.20}$$

With these identities, (8.20) and (8.14), established, consider again the basic equations of motion. They are expressed in the matrix equation form as

$$M\ddot{X} + C\dot{X} + KX = F \tag{8.21}$$

We now attempt to express the solution vector for this equation as a linear sum of the modal vectors as follows:

$$\begin{bmatrix} X_N \\ \cdot \\ \cdot \\ \cdot \\ X_1 \\ \theta \end{bmatrix} = \begin{bmatrix} e_{1,1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ e_{N+1,1} \end{bmatrix} p_1 + \begin{bmatrix} e_{1,2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ e_{N+1,2} \end{bmatrix} p_2 + \cdots \begin{bmatrix} e_{1,N+1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ e_{N+1,N+1} \end{bmatrix} p_{N+1} \tag{8.22}$$

This can be expressed more compactly as

$$X = QP$$

where Q was as previously defined and P is defined as the vector $P = \left\{ p_1, p_2, \ldots p_{N+1} \right\}$ with elements p_i. Substituting in for X in (8.21) its equivalent QP results in

$$M Q\ddot{P} + CQ\dot{P} + KQP = F \tag{8.23}$$

Premultiplying by Q^T yields

$$Q^T MQ\ddot{P} + Q^T CQ\dot{P} + Q^T KQP = Q^T F \tag{8.24}$$

Now for the undamped case this equation reduces to

$$Q^T MQ\ddot{P} + Q^T KQP = Q^T F \tag{8.25}$$

Applying the relationships (8.14) and (8.20) yield

$$\ddot{P} + \Lambda P = Q^T F \tag{8.26}$$

This matrix equation is a decoupled set of simultaneous linear differential equations due to the fact that Λ is diagonal. Thus, the values of p_i can be determined by the set of equations

$$\ddot{p}_1 + \omega_1^{\,2} p_1 = e_1^{\,T} F \tag{8.27}$$

$$\cdots\cdots\cdots\cdots$$

$$\ddot{p}_{N+1} + \omega_{N+1}^2 \, p_{N+1} = e_{N+1}^T F$$

It is to be observed that the right hand side of each of the above equations is a scalar resulting from the product of N+1 by one and one by N+1 vectors. Given the p_i values that result from a solution of this set of independent equations the values of X_i's can be determined according to equation (8.22). The p_i values that represent the solution of these equations are, of course, solutions of ordinary differential equations with time as the independent variable and a forcing function ($e_i^T F$) that is a scalar multiple of the ground motion acceleration. Inasmuch as ground motion acceleration due to seismic disturbance is of erratic, random character, the determination of p_i values (which are a function of time) and the vector combination of modal shapes weighted by these values to obtain displacements requires an extensive computerized solution. For typical problems it is customary to follow, therefore, the approach suggested by Clough (8) and others.

It is suggested by Clough that a good estimate of the maximum value of structural displacement is obtained by combining the modal vector displacements (i.e. the e_i p_i values where e_i is a vector) in a suitable manner. For a small number of modes (two or three) a absolute sum is recommended as the most appropriate combination. However, where the number of modes is large a square root of the sum of the squares of modal maxima is suggested.

Damping can be considered by retaining the $Q^T C Q\dot{P}$ term of equation (8.24). However, to achieve decoupling this $Q^T C Q\dot{P}$ term must now also be diagonal. Otherwise it can be seen that the ith new equation of the set analogous to the (8.27) set would contain $\dot{p}_j$ terms where i is not equal to j. Mathematically this is equivalent to requiring that

$$Q^T C Q = D \tag{8.28}$$

where the matrix D is of order N+1 with diagonal form.

Since the diagonal terms of this D matrix are numerical coefficients they can be expressed as the ω for the corresponding mode times a factor, 2f, where f is numerical constant selected such as to make $2\omega_i f_i$ equal to the corresponding D matrix entry. In such a manner each diagonal entry of the D matrix (all off diagonal elements are zero) is replaced by a numerically equivalent $2\omega_i f_i$ term with f so adjusted to make $2\omega_i f_i = d_{ii}$. We have then an array of diagonal $2\omega_i f_i$'s, one for each mode.

From forced harmonic vibration testing of structures it is possible to observe both structural respones during forced vibration and decay of the structure's vibrational amplitude after termination of force application. By such testing it is possible, at least in principle, to obtain frequencies and modal damping values for the undamped modes of small damped structures. If such modal frequencies and modal damping values, then, are obtainable by test we have, in effect, achieved a knowledge of both the ω_i^2 and $2\omega_i f_i$ values for the equations

$$\ddot{P} + D\dot{P} + \Lambda P = Q^T F \tag{8.29}$$

Note that when D, Λ, Q^T and F matrices are known this equation set yields directly a set of p_i values.

By having a knowledge of these matrices such as is, in principle, obtainable from the dynamic testing described above, one can also proceed to determine fundamental structural damping coefficients that are usable in an analog computer to determine the response of the same structure under dynamic loading. This can be achieved from the following considerations. Premultiplying (8.29) by $Q^{T^{-1}}$ yields

$$Q^{T^{-1}}\ddot{P} + Q^{T^{-1}}D\dot{P} + Q^{T^{-1}} P = F \tag{8.30}$$

Comparing (8.30) and (8.23) it is obvious that the following equalities must be valid if the equations of (8.30) are to be reducible to their original form

$$Q^{T^{-1}} = MQ \tag{8.31}$$

and

$$Q^{T^{-1}} D = CQ \tag{8.32}$$

Substituting (8.31) into (8.32) yields

$$MQD = CQ$$

This matrix equation is an array of $(N+1)^2$ simultaneous equations in which the C_{ij}'s are the only unknowns. Postmultiplying by Q^T yields

$$C = MQDQ^{-1}$$

Thus, the C_{ij} elements of the C matrix are then obtained by equating corresponding coefficients.

It is to be emphasized that the above procedures are valid only where damping is small and/or where the damping coefficients have a special relationship to the mass and/or stiffness coefficients. The significance of this statement can be more fully appreciated by considering the special cases where the C matrix elements are proportional to the mass or stiffness matrix elements or to a linear combination of the elements of both of them.

First, assume that

$$C = \gamma M \tag{8.33}$$

where γ is a proportionality factor. Using the equations of motion (8.23) and multiplying by Q^T yields

$$Q^T M Q \ddot{P} + Q^T C Q \dot{P} + Q^T K Q P = Q^T F \tag{8.34}$$

Substituting γM for C and recalling that $Q^T M Q = I$ yields

$$\ddot{P} + \gamma I \dot{P} + \Lambda P = Q^T F \tag{8.35}$$

The modal equations then become

$$\ddot{p}_1 + \gamma \dot{p}_1 + \omega_1^2 p_1 = e_1^T F \tag{8.36}$$

$$* * * * * * * * * * * * *$$

$$\ddot{p}_{N+1} + \gamma \dot{p}_{N+1} + \omega_{N+1}^2 p_{N+1} = e_{N+1}^T F$$

Observing that γ is a constant and, also, that D can be substituted for γI if all $2\omega_i f_i$'s equal γ, it is seen that requiring $C = \gamma M$ is equivalent to specifying the fraction of critical damping in each mode to be inversely proportional to the modal frequency of that mode. Noting that $C = \gamma M$ also requires the form of the C and M matrices to be identical, it can also be seen by going back to basic equation formulation that this is achievable only when system damping is absolute. That is, the system is idealized by assigned damping effects only between the individual masses and the foundation.

In an analogous manner, it can be shown that requiring

$$C = \beta K \tag{8.37}$$

where β is a proportionality factor is equivalent to both requiring modal damping to be directly proportional to system frequencies and specifying the physical assignment of dampers to be such as to assign damping to the interfloor motion of masses (relative damping).

From the above paragraphs it can be seen that specification of damping as being of a proportional type is mathematically appealing. However, for damping, in structures most experimental evidence indicates it to be frequency independent rather than being directly or inversely proportional to frequency. In addition, the problem that is the subject of this research investigation is one where the foundation stiffness can be considerably smaller than structural stiffness whereas the foundation damping can be considerably greater than the structural damping. Such a situation is extremely non-proportional.

Where the foundation damping is high with respect to the structural damping and where, for the specific problem, the foundation mass is, also, large with respect to structural mass, one might be tempted to apply a proportional damping formulation. However, the associated implication that modal damping is inversely proportional to modal frequency shows this approach to be of questionable validity, also.

B. General Solution for Two Free Mass System

The set of equations of motion for the general two free mass system have the following form as developed in section V.

$$M\ddot{X} + C\dot{X} + KX = F \tag{8.38}$$

In finding the free vibrational periods and modes, the forcing function and damping matrices are disregarded and, as indicated in the previous section, a solution of the form $X_i = y_i e^{i\omega t}$ and $\theta = \Phi e^{i\omega t}$ is sought. Making the substitution yields

$$\begin{aligned} &-\omega^2(M_2y_2 + M_2H_2\Phi) + K_2y_2 - K_2y_1 = 0 \\ &-\omega^2(M_1y_1) + (K_1 + K_2)y_1 - K_2y_2 = 0 \\ &-\omega^2(M_2H_2y_2 + M_2H_2{}^2\Phi + I_0\Phi) + K_r\Phi = 0 \end{aligned} \tag{8.39}$$

or, rearranging,

$$\begin{aligned} &(-M_2\omega^2 + K_2)y_2 - (K_2)y_1 - (\omega^2M_2H_2)\Phi = 0 \\ &-K_2y_2 + (-\omega^2M_1 + K_1 + K_2)y_1 = 0 \\ &-\omega^2M_2H_2y_2 + [-\omega^2(M_2H_2{}^2 + I_0) + K_r]\Phi = 0 \end{aligned} \tag{8.40}$$

Equating the determinant of the coefficients equal to zero yields the auxilary equation

$$(-\omega^2M_2 + K_2) - (\omega^2M_1 + K_1 + K_2)Y + K_2{}^2Y - (\omega^4M_2{}^2H_2{}^2)(-\omega^2M_1 + K_1$$

$$+ K_2) = 0 \tag{8.41}$$

where $Y = -\omega^2 M_2 H_2{}^2 - \omega^2 I_0 + K_r$

Expanding this expression yields

$$\begin{aligned}(-M_1M_2I_0)\omega^6 &+ M_1M_2K_r + K_1M_2I_0 + K_2M_2I_0 + K_2M_1M_2H_2{}^2 \\ &+K_2M_1I_0)\omega^4 - K_rK_1M_2 + K_rK_2M_2 + K_rK_2M_1 + K_1K_2M_2H_2{}^2 \\ &- K_1K_2I_0)\omega^2 + K_1K_2K_r = 0\end{aligned} \tag{8.42}$$

Note that the frequency equation is of the third order in ω^2. The calculation of the ω^2 values and the associated eigenvectors follows in a straight forward manner.

If I_0 is neglected as being small, as is often times the case, the equation reduces to

$$\begin{aligned}(K_rM_1M_2 &+ K_2M_1M_2H_2{}^2)\omega^4 + (-K_1K_rM_2 + K_2K_rM_1 + K_2K_rM_2 \\ &+ K_1K_2M_2H_2{}^2)\omega^2 + K_1K_2K_r = 0\end{aligned} \tag{8.43}$$

It can be seen from the above frequency equation that neglecting local moments of inertia of system masses reduces the number of degrees of freedom of the system by one when rocking motion is considered. This is not the case for a no rocking situation, however, since θ does not enter into the translation-only development.

The significance of this reduction in degree-of-freedom can be seen using the two mass case as an example. The equations of the rocking-only two mass system are

$$M_2\ddot{X}_2 + M_2H_2\ddot{\theta} + M_2\ddot{X}_0 + C_2\dot{X}_2 + K_2X_2 = 0 \tag{8.44}$$

$$M_2H_2\ddot{X}_2 + M_2H_2\ddot{X}_0 + M_2H_2^2\ddot{\theta} + I_0\ddot{\theta} + C_r\dot{\theta} + K_r\theta = 0$$

Neglecting damping and I_0 terms and solving for $\ddot{\theta}$ in the second equation yields

$$\ddot{\theta} = -\left(\frac{M_2H_2\ddot{X}_2 + M_2H_2\ddot{X}_0 + K_r\theta}{M_2H_2^2}\right) \tag{8.45}$$

Substituting for $\ddot{\theta}$ in the first equation of the previous set yields

$$M_2\ddot{X}_2 - M_2H_2\left(\frac{\ddot{X}_2}{H_2} + \frac{\ddot{X}_0}{H_2} + \frac{K_r}{M_2H_2^2}\theta\right) + M_2\ddot{X}_0 + K_2X_2 = 0 \tag{8.46}$$

and, rearranging,

$$X_2 = \frac{K_r}{K_2H_2}\theta \tag{8.47}$$

$$\theta = \frac{K_2H_2}{K_r}X_2 \tag{8.48}$$

Making the substitutions $X_i = y_i e^{i\omega t}$ and $\theta = \Phi e^{i\omega t}$ as before yields the alternate equivalent relationship

$$y_2 = \frac{K_r}{K_2H_2}\Phi$$

Thus it is seen that the y_2 motion (in an undamped system) and the Φ motion are directly related when I_0 is neglected. For the specific case just used (the undamped two free mass no translation case) when I_0 is neglected the system is, in reality, a one degree-of-freedom system with the relationship between θ

and X_2 as previously expressed.

By resubstitution back into the first equation of (8.44), it can be shown that there results the differential equation

$$\ddot{X}_2 + \frac{\omega^2}{\left[1 + \frac{H_2^{\,2}K_2}{K_r}\right]^2} X = - \frac{\ddot{X}_0}{\left[1 + \frac{H_2^{\,2}K_2}{K_r}\right]} \tag{8.49}$$

The natural frequency of the one degree-of-freedom fixed base system retains its one degree-of-freedom character but the natural frequency and the effective ground acceleration have been modified by the factor

$$\frac{1}{\left[1 + \frac{K_2H_2^{\,2}}{K_r}\right]}$$

For this one degree-of-freedom system the change in system period as influenced by foundation stiffness and structure height is shown in figure seventeen.

C. Modal Analysis Solution for Multimass Problem

The modal analysis method previously discussed is used in this part to evaluate the influence of foundation rotational and translational stiffness on structural response for a specific case as an example of the method. The results from the example along with those from a couple of other cases were also useful in checking the validity of the computer program developed to automate the calculations.

Consider the general two free mass problem. For the general case it has been previously formulated (see equation

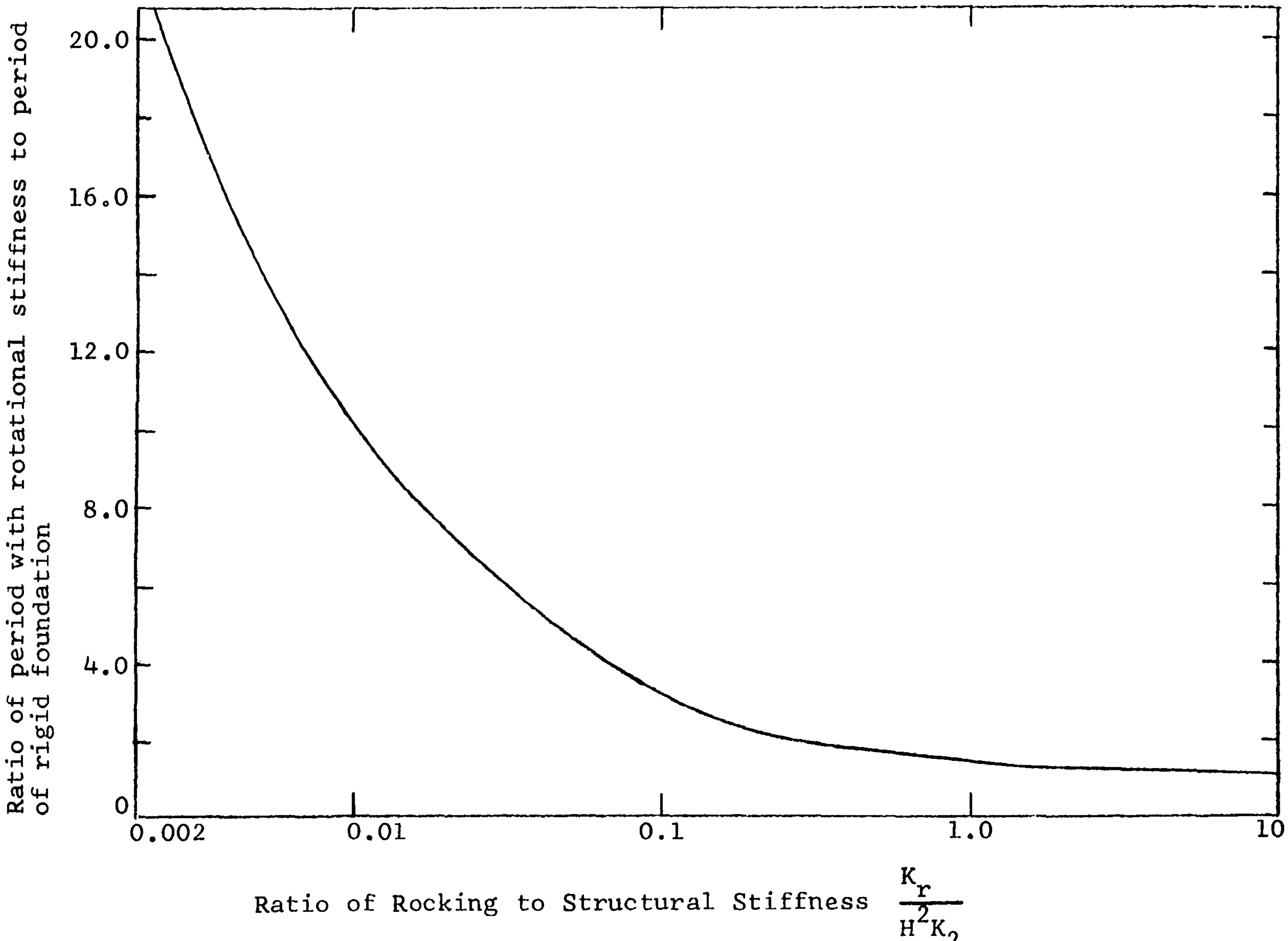


Figure 17. Variation in structural period with characteristic parameter

5.8) as

$$\begin{bmatrix} M_2 & 0 & M_2H_2 \\ 0 & M_1 & 0 \\ M_2H_2 & 0 & I_0+M_2H_2{}^2 \end{bmatrix} \begin{bmatrix} \ddot{X}_2 \\ \ddot{X}_1 \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} K_2 & -K_2 & 0 \\ -K_2 & K_1+K_2 & 0 \\ 0 & 0 & K_r \end{bmatrix} \begin{bmatrix} X_2 \\ X_1 \\ \theta \end{bmatrix}$$

$$= - \begin{bmatrix} M_2 & 0 & 0 \\ 0 & M_1 & 0 \\ 0 & 0 & M_2H_2 \end{bmatrix} \begin{bmatrix} \ddot{X}_0 \\ \ddot{X}_0 \\ \ddot{X}_0 \end{bmatrix} \qquad (8.50)$$

or, making the standard substitution and neglecting the forcing function, (8.51)

$$\omega^2 \begin{bmatrix} M_2 & 0 & M_2H_2 \\ 0 & M_1 & 0 \\ M_2H_2 & 0 & I_0+M_2H_2{}^2 \end{bmatrix} \begin{bmatrix} y_2 \\ y_1 \\ \Phi \end{bmatrix} = \begin{bmatrix} K_2 & -K_2 & 0 \\ -K_2 & K_1+K_2 & 0 \\ 0 & 0 & K_r \end{bmatrix} \begin{bmatrix} y_2 \\ y_1 \\ \Phi \end{bmatrix}$$

Take now as an example case a type I structure with both rocking and translational foundation freedom and with a fairly stiff foundation. The computer output for this example is shown in Appendix A. Substituting the values for mass and stiffness as given in Appendix A yields (8.52)

$$\omega^2 \begin{bmatrix} 591 & 0 & 76900 \\ 0 & 1880 & 0 \\ 76900 & 0 & 25.0\text{x}10^6 \end{bmatrix} \begin{bmatrix} y_2 \\ y_1 \\ \Phi \end{bmatrix} = \begin{bmatrix} 1.58 & -1.58 & 0 \\ -1.58 & 4.40 & 0 \\ 0 & 0 & 10{,}000 \end{bmatrix}$$

$$\begin{bmatrix} y_2 \\ y_1 \\ \Phi \end{bmatrix}$$

Obtaining the inverse for the mass matrix and premultiply the equation through by this matrix yields

$$\omega^2 \begin{bmatrix} y_2 \\ y_1 \\ \Phi \end{bmatrix} = \begin{bmatrix} 4450 & -4450 & -86{,}900 \\ -845 & 2350 & 0 \\ -13.75 & -13.75 & 667 \end{bmatrix} \begin{bmatrix} y_2 \\ y_1 \\ \Phi \end{bmatrix}$$

$$\begin{aligned} &\omega_1^2 = 356.92 \quad \lambda_1 = 3.008 \\ &Y_1 = \{0.92051,\ 0.39003,\ 0.02346\} \\ &\omega_2^2 = 1287.35 \quad \lambda_2 = 5.713 \\ &Y_2 = \{0.78184,\ 0.62377,\ -0.00349\} \\ &\omega_3^2 = 5820.60 \quad \lambda_3 = 12.148 \\ &Y_3 = \{0.97163,\ -0.2347,\ -0.00321\} \end{aligned} \tag{8.53}$$

Using the relationship $Y_i^T M Y_i = L_i^2$ (a scalar) from equation (8.11) a matrix of L_i^2 values is computed. It is

$$\begin{bmatrix} 439.7297 & 0.0004 & 0.0000 \\ 0.0005 & 977.3923 & -0.0019 \\ -0.0007 & -0.0001 & 17870.3700 \end{bmatrix} \tag{8.54}$$

where the value in the first row, first column is $Y_3^T M Y_3 = L_3^2$, the value in the second row, second column is $Y_2^T M Y_2 = L_2^2$ and the value in the third row, third column is $Y_1^T M Y_1 = L_1^2$. The off diagonal elements are formed in an analogous manner. Note that the off diagonal elements are of zero value to within the accuracy of the calculation.

Dividing each vector of (8.53) by its corresponding normalization factor yields

$$e_1 = \{0.00689,\ 0.00292,\ 0.00018\}$$

$$e_2 = \{0.02500,\ 0.01995,\ -0.00011\} \qquad (8.55)$$

$$e_3 = \{0.04636,\ -0.01119,\ -0.00015\}$$

Referring to equation 8.25 and noting that the forcing function matrix is expressed as:

$$F = \{\ M_2\ddot{X}_0 \quad M_1\ddot{X}_0 \quad M_2H_2\ddot{X}_0\ \} \qquad (8.56)$$

The forcing function for the first decoupled participation factor equation is expressible as

$$[0.00689 \quad 0.00292 \quad 0.00018]\begin{bmatrix} 591 \\ 1880 \\ (591)(130) \end{bmatrix} \ddot{X}_0 \qquad (8.57)$$

or, performing the indicated multiplication,

$$e_1{}^T F = [(0.00680)(591) + (0.00292)(1880) + (591)(130)$$
$$(0.00018)]\ddot{X}_0 = 23.051\ddot{X}_0$$

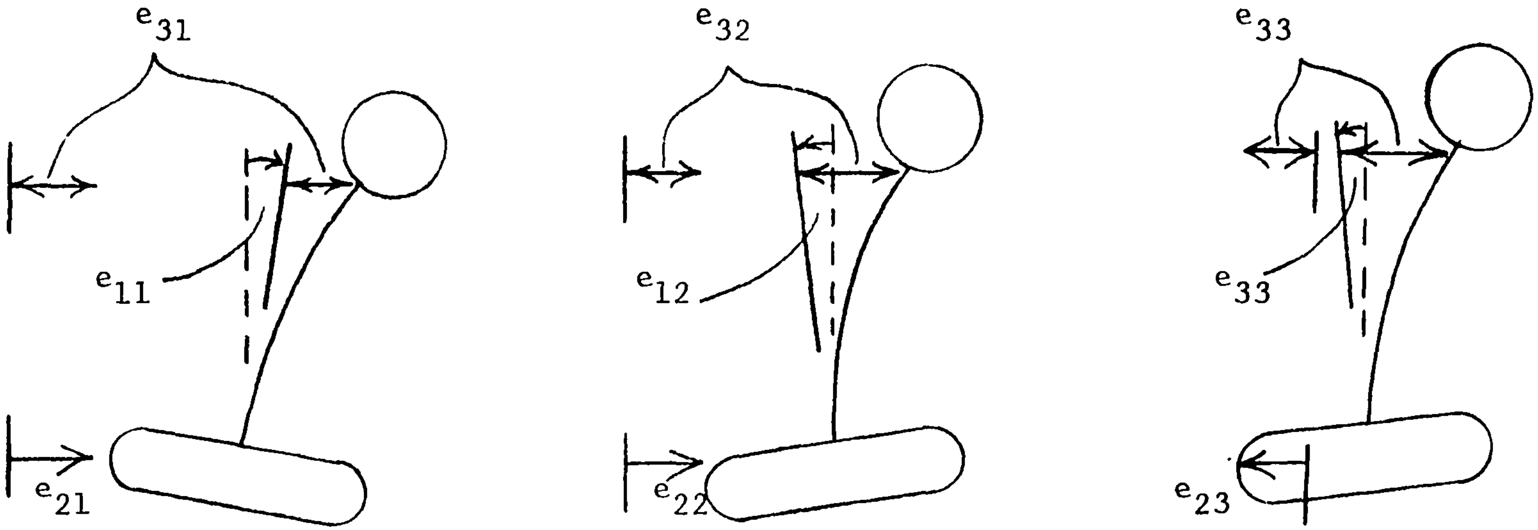
Likewise,

$$e_2{}^T F = 43.706\ddot{X}_0$$

and

$$e_3{}^T F = 5.423\ddot{X}_0$$

Modal damping percentages appropriate to the type of structure are now identified and the corresponding response spectrum curves selected. The maximum structural distortion in each mode (which is the same as the original X_2 coordinate of each mode)



First Mode Second Mode Third Mode

Figure 18. Mode shapes for two free mass example problem

is then obtained by securing from the selected spectrum (in this case the zero damping curve of figure thirteen) the structural displacement value corresponding to the frequency of the mode, multiplying this value by the corresponding $e_i^T F$ coefficient, and multiplying the resulting product by the relative modal structural displacement. This procedure yields the following maximum modal structural distortions for each mode for the zero modal damping case.

First mode distortion:

(0.105)(23.051)(0.0068 - 0.00292) = 0.00989

Second mode distortion:

(0.049)(43.706)(0.0250 - 0.01995) = 0.01110

Third mode distortion:

(0.021)(5.423)(0.04636 + 0.0119) = 0.00649

Taking the sum of the absolute values of these modal distortions yields a total structural distortion of 0.02735 feet. Computing a square root of the sum of the squares results in a total structural distortion of 0.01615 feet.

In summary, in this section (Section VIII C) a specific example of the general two mass problem was solved step-by-step to illustrate the procedures used. The general FORTRAN listing to solve this problem by the digital computer is shown in Appendix A. A typical output results sheet, specifically the one for the particular example chosen to illustrate the solution procedure, is also shown in Appendix A.

IX. RESULTS

A. General

In this section the results of parameter studies involving variation in foundation stiffness and damping are presented. As indicated in the program of investigation a series of five containment structures, representing the range of types and sizes of nuclear containment structures currently being placed into service, was selected and structural parameters suitable to them were calculated from available data. Investigation was made of the effect on modal frequency and structural displacement and response as type of foundation, foundation stiffness, foundation damping and structural mass and moment of inertia were varied.

The results of the structural deformation computations are expressed in the form of structural displacements (defined as the maximum relative displacement between superstructure levels, i.e. maximum X_2 in figure seven) and structural response (defined as the ratio of maximum relative displacement to the displacement of the same structural model with a fixed base).

B. Foundation Stiffness Influence on Modal Frequencies

The modal frequencies of undamped free vibrational modes of two free mass structures have been determined for the I^1 equated top mass cases studied in this investigation. For review of the method used to compute these frequencies the reader should refer to the previous chapter. As typical of the influence that foundation stiffness variations have on the modal frequencies of foundation-structure lumped mass systems the modal frequencies (λ's) of such systems are graphed as a function of the ratio of foundation-to-structural stiffness (K_1/K_2 or K_r/K_2 as applicable). The change that occurs in these modal frequencies for a type I structure with either a translational or rotational foundation for a change in foundation stiffness is shown in figure nineteen. The corresponding change that occurs in the modal frequencies of the companion taller type II structure is shown in figure twenty.

The effect of variation in foundation stiffness on the combined constant K_r and variable K_r structures has also been calculated. For the combined type I structures the results are graphed in figure twenty-one. For the combined type II structures the results are correspondingly graphed in figure twenty-two.

The use in these figures and subsequently of the expression "Rotational Structure" refers to a structure where the foundation is translationally very stiff and whose base translates along with the ground translation. There is, however, rotational or rocking action. The use of the expression "Translational Structure" refers to the opposite case where the structure's foundation is rotationally very stiff and the only motion in the system is one of translation. The "Combined Structure" term is used to denote a structure where both foundation translation and rotation can occur. Two subcases of the "Combined Structure" category have been investigated. They are the situations a) where the rotational stiffness is constant (referred to as the constant K_r case) and b) where the rotational stiffness is proportional to the translational stiffness (called the variable K_r case).

In the case of combined constant K_r structures a value of 10×10^6 ft-kips was selected as being an appropriate value for the rotational stiffness of the type I, II, III and IV structures. A value of 30×10^6 ft-kips was selected as the K_r value for the type V structure. The above values were chosen on the basis of their previous usage in containment structure designs that included pile foundations.

For the case of combined variable K_r structures the K_r chosen was one that was 3550 times the K_1 value of the foundation for type I, II, III and IV structures. For the type V combined variable K_r structure a value of 6120 times the K_1 of the foundation was selected. These values were selected to be representative of the average relationship between soil translational and rotational stiffnesses (1).

C. Foundation Stiffness Influence on Structural Response

1. General

Using the modal analysis approach described in section VIII and the undamped linear approximation to the El Centro response spectrum shown in figure thirteen, absolute sum of modal maxima and square root of the sum of the squares of modal maxima structural displacement and structural response values were calcu-

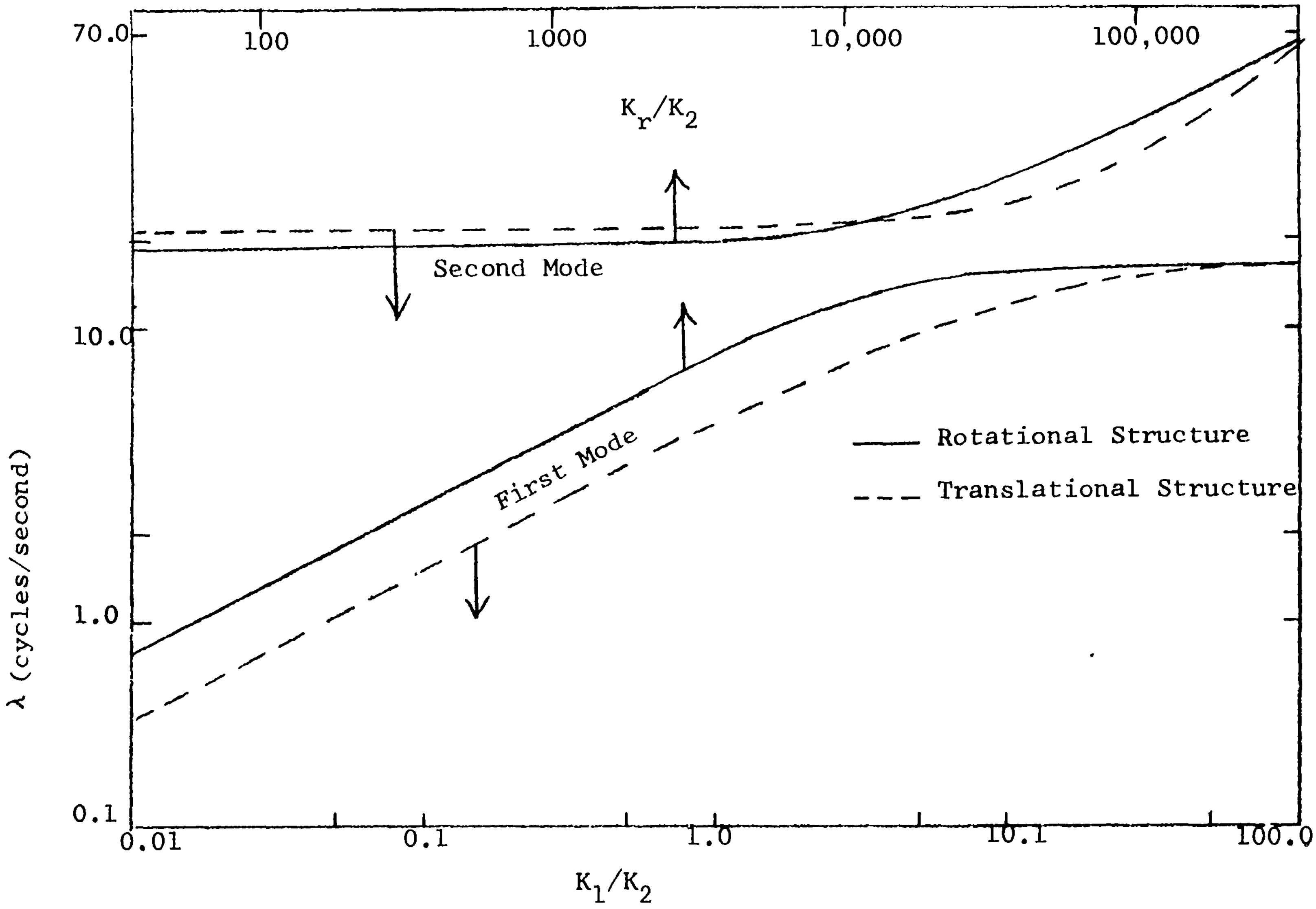


Figure 19. Alteration in structural vibrational frequencies by variation in foundation stiffness for type I structures

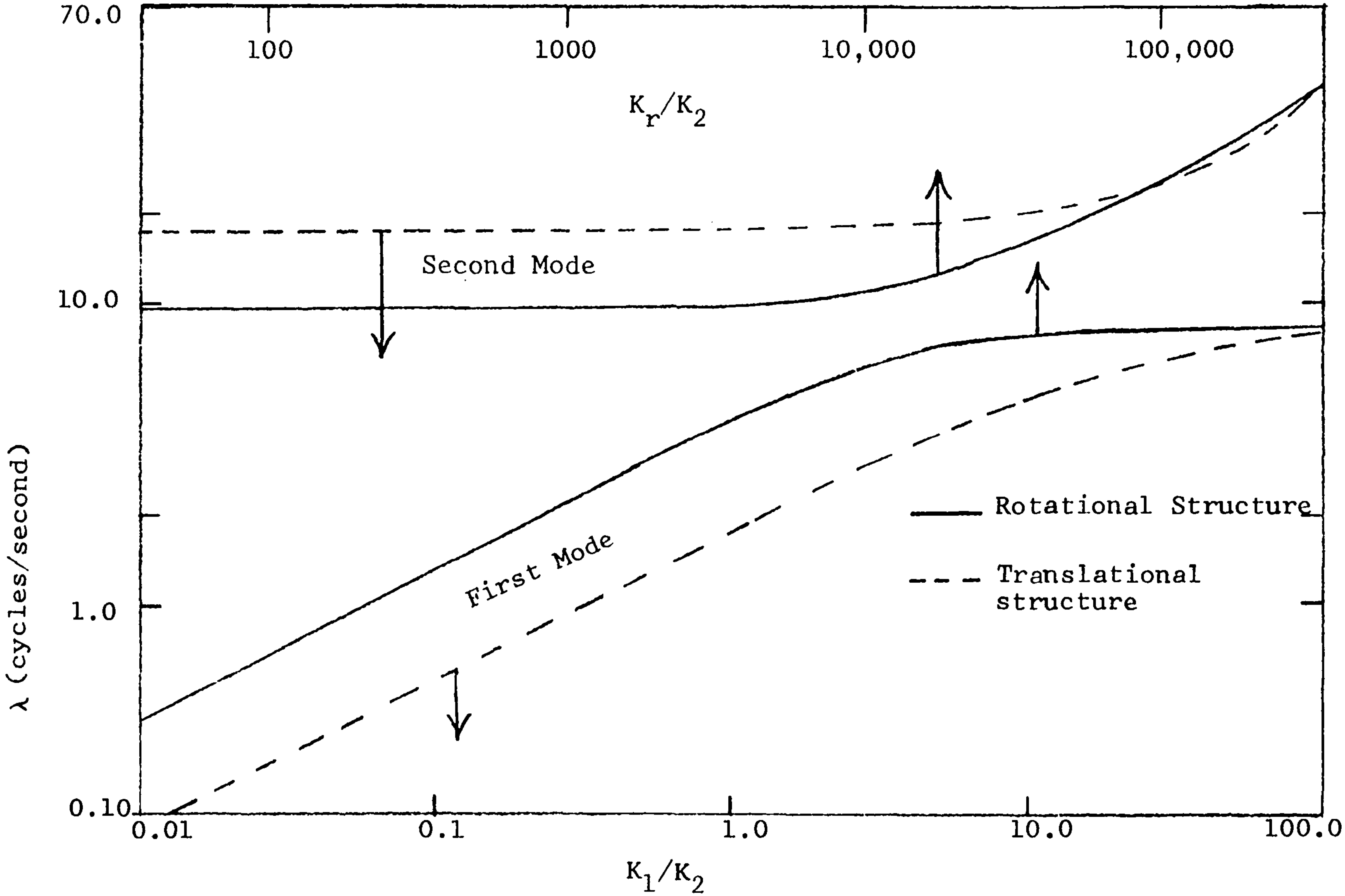


Figure 20. Alteration in structure vibrational frequencies by variation in foundation stiffness for type II structures

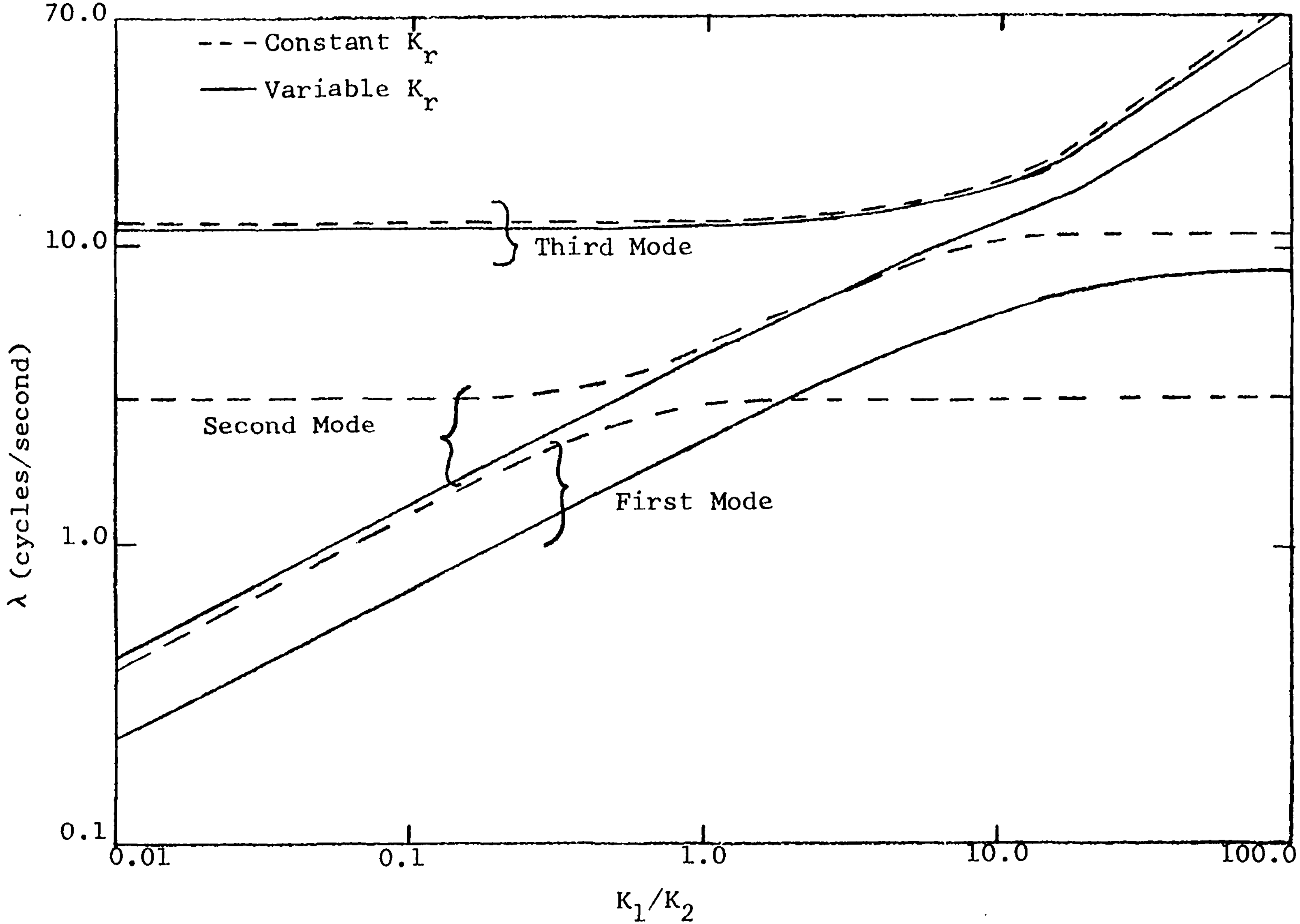


Figure 21. Alteration in structure vibrational frequencies with variation in foundation stiffness for type I combined structures

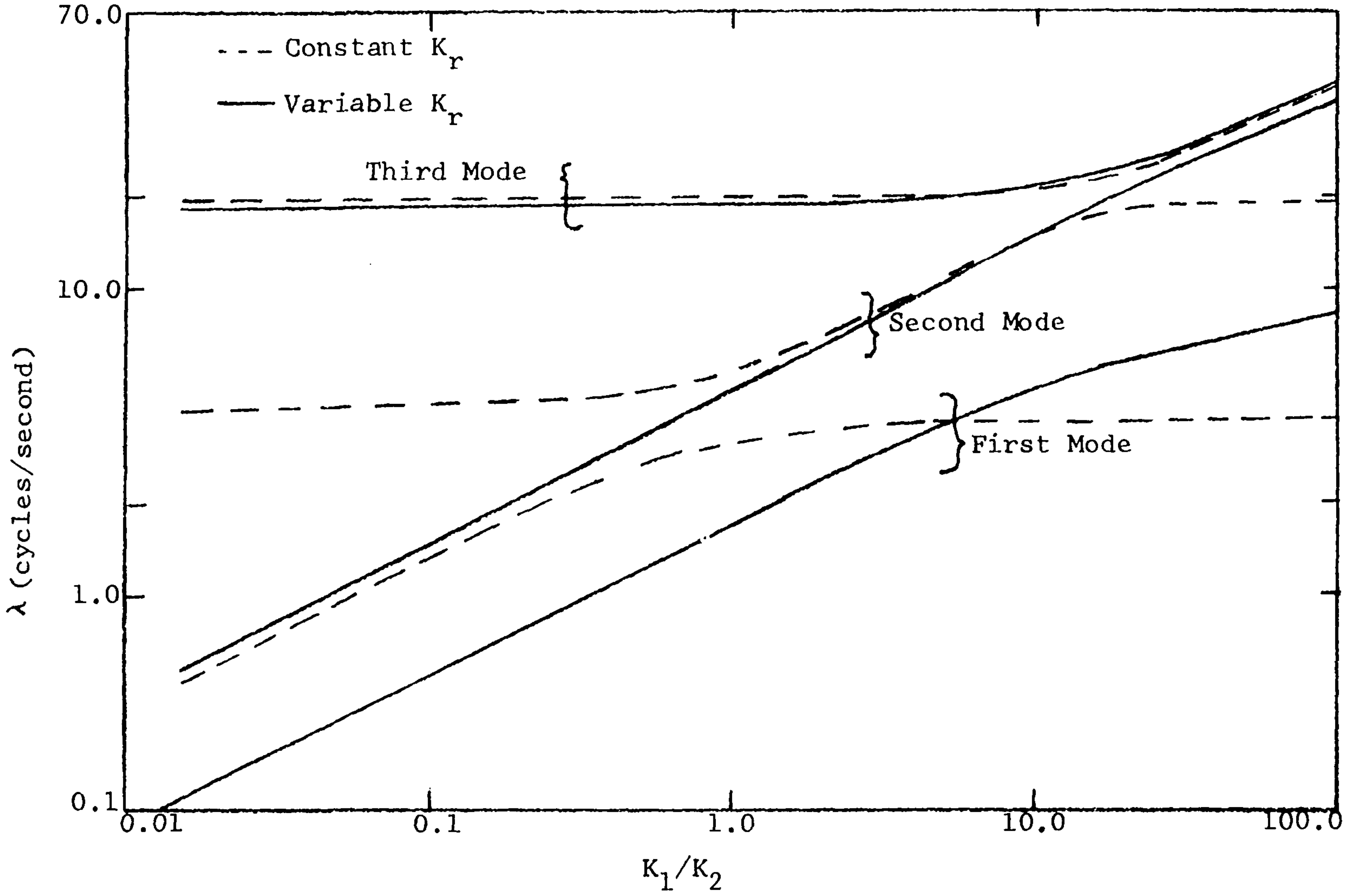


Figure 22. Alteration in structure vibrational frequencies with variation in foundation stiffness for type II combined structures

lated by a digital computer program (see Appendix A) for a number of structural types and variations in foundation freedom characteristic. For each specific structure type (Type I, II, III, IV or V) and foundation characteristic (translational, rotational or combined) a series of fourteen foundation stiffness values were selected. These foundation stiffness values were so spaced as to cover the entire range of possible useful foundation stiffnesses. Individual maxima referred to above were then obtained for each of these selected stiffness values.

The results of these computations in the form of a sum of modal maxima structural displacement and a sum of modal maxima structural response are graphed as a function of the appropriate stiffness ratio (K_1/K_2 or K_r/K_2). The graphed results are in the form of lines connecting data points. No attempt was made to smooth or fit the resulting data to a curve.

An outgrowth of the two free mass analytical investigation of section VIII was the appearance, for rocking motion, of the parameter K_r/H^2K_2. Since occasionally results obtained for the rocking action of structures are expressed in terms of this parameter, it may be useful to give at this point the relationship between this parameter and the K_r/K_2 parameter used in this study. For the type I and III structure a 10^5 value of K_r/K_2 corresponds to a K_r/H^2K_2 value of about 6.0; for the type II and IV structures it corresponds to a K_r/H^2K_2 of about 1.5; and for a type V structure it corresponds to a K_r/H^2K_2 value of about 2.0. For the type I and III structures H_2 is 130 feet.

For the type II and IV structures H_2 is 250 feet. Likewise, for the type V structure H_2 is 140 feet.

2. Structures with translational freedom

The influence of foundation stiffness on the response of structures with translational foundations is shown in figures twenty-three through twenty-seven. In figure twenty-three the maximum structural displacement developed in a type I structure when subjected to an El Centro type ground motion is graphed as a function of the ratio of foundation translational stiffness to structural lateral stiffness. The four curves shown in the figure refer to four different assumptions on the partition of mass in the structure, the details for selection of which are described in section VI. In figure twenty-four essentially the same information is shown. This figure relates the stiffness ratio (K_1/K_2) to structural response rather than structural displacement.

In figures twenty-five and twenty-six the same sets of ordinates and mass assumptions are used to describe the effect of ground motion on a type II containment structure. In figure twenty-seven the structural response of three containment structure types, types III, IV and V, are graphed as a function of stiffness ratio. For these structures no attempt was made to discern the effect of various mass partition assumptions. The I^1 equated top mass partition was used exclusively.

3. Structures with rotational freedom

The influence of foundation stiffness on the response of

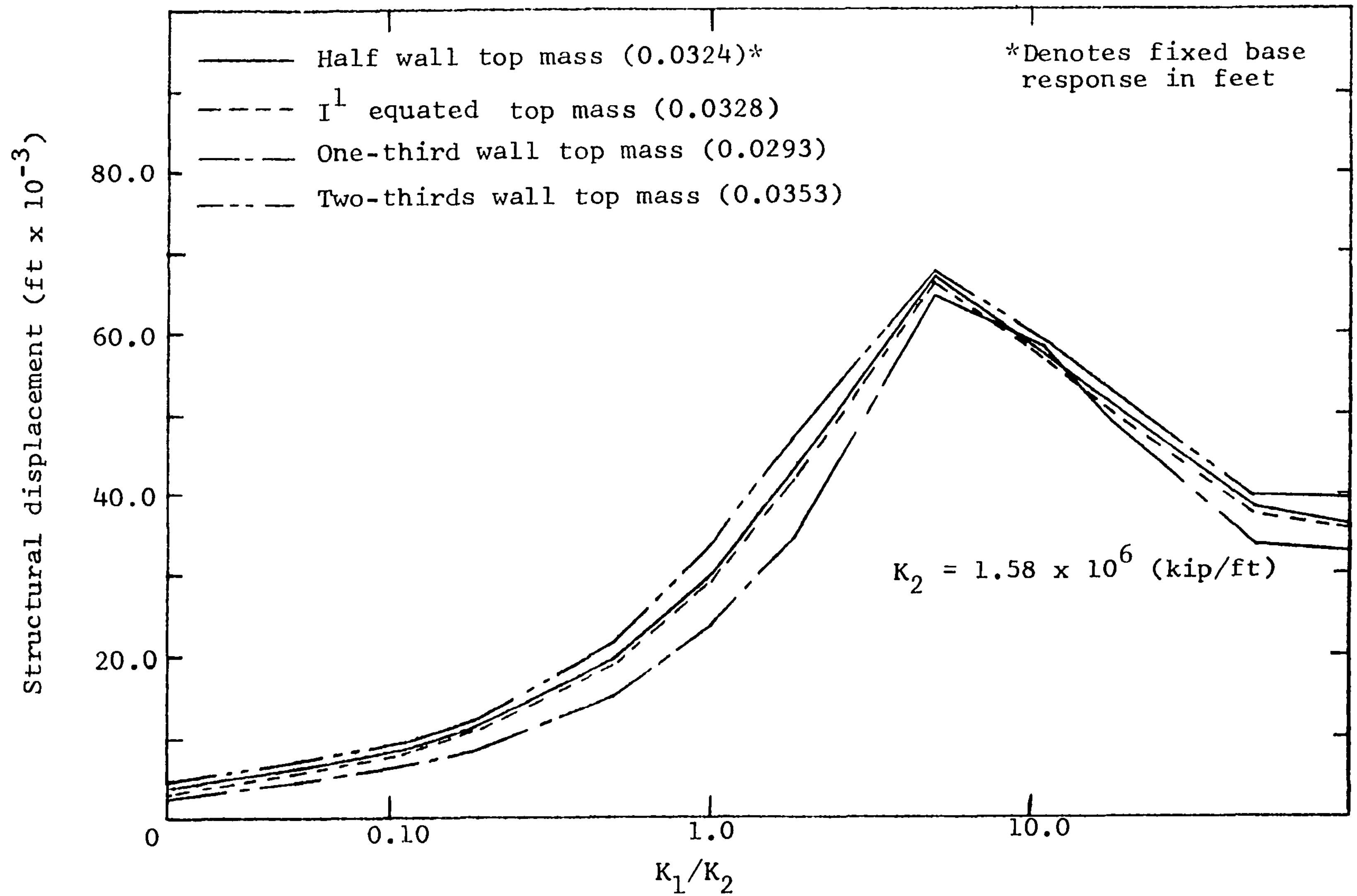


Figure 23. Type I translational structure displacement as a function of stiffness ratio

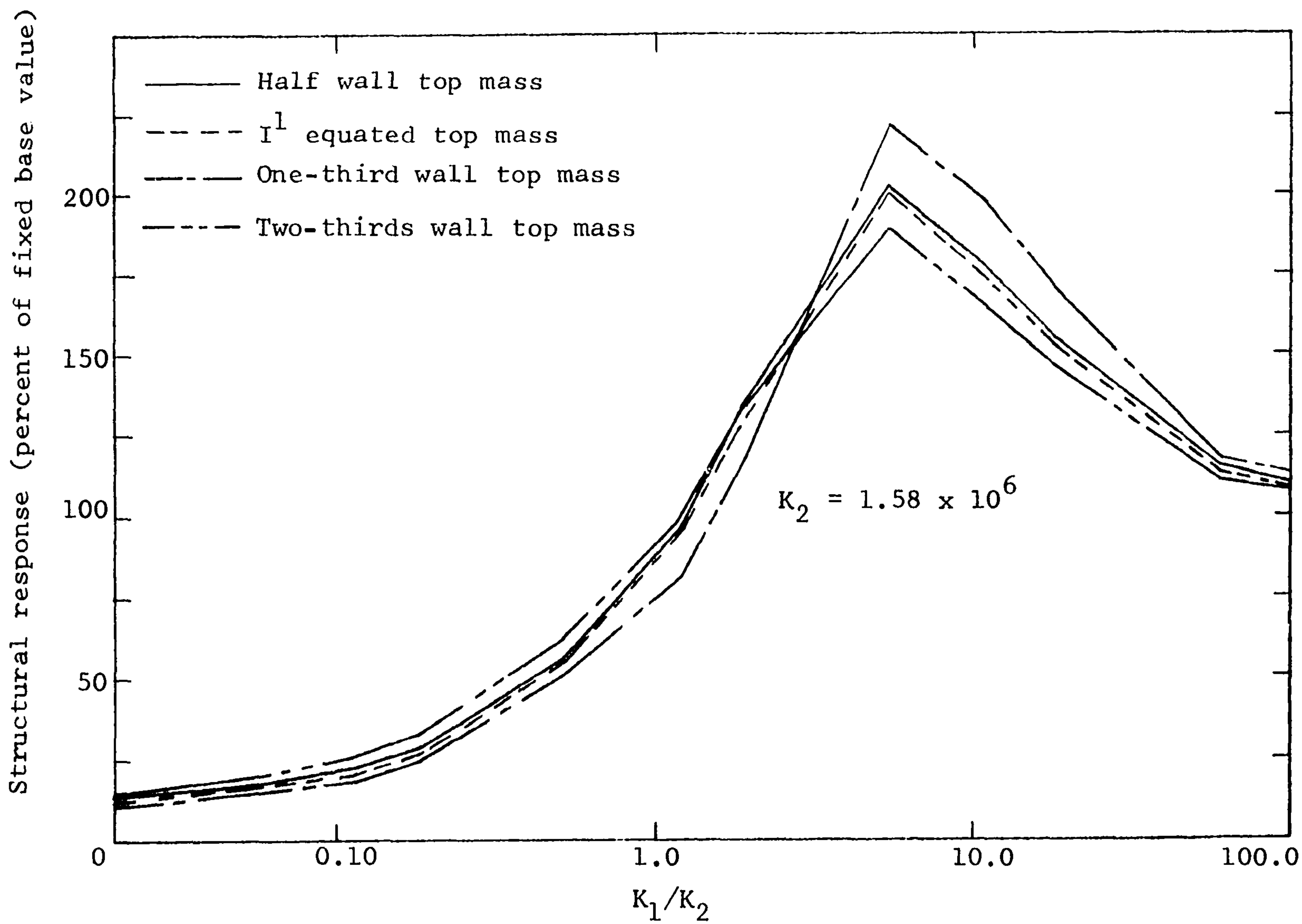


Figure 24. Type I translational structure response as a function of stiffness ratio

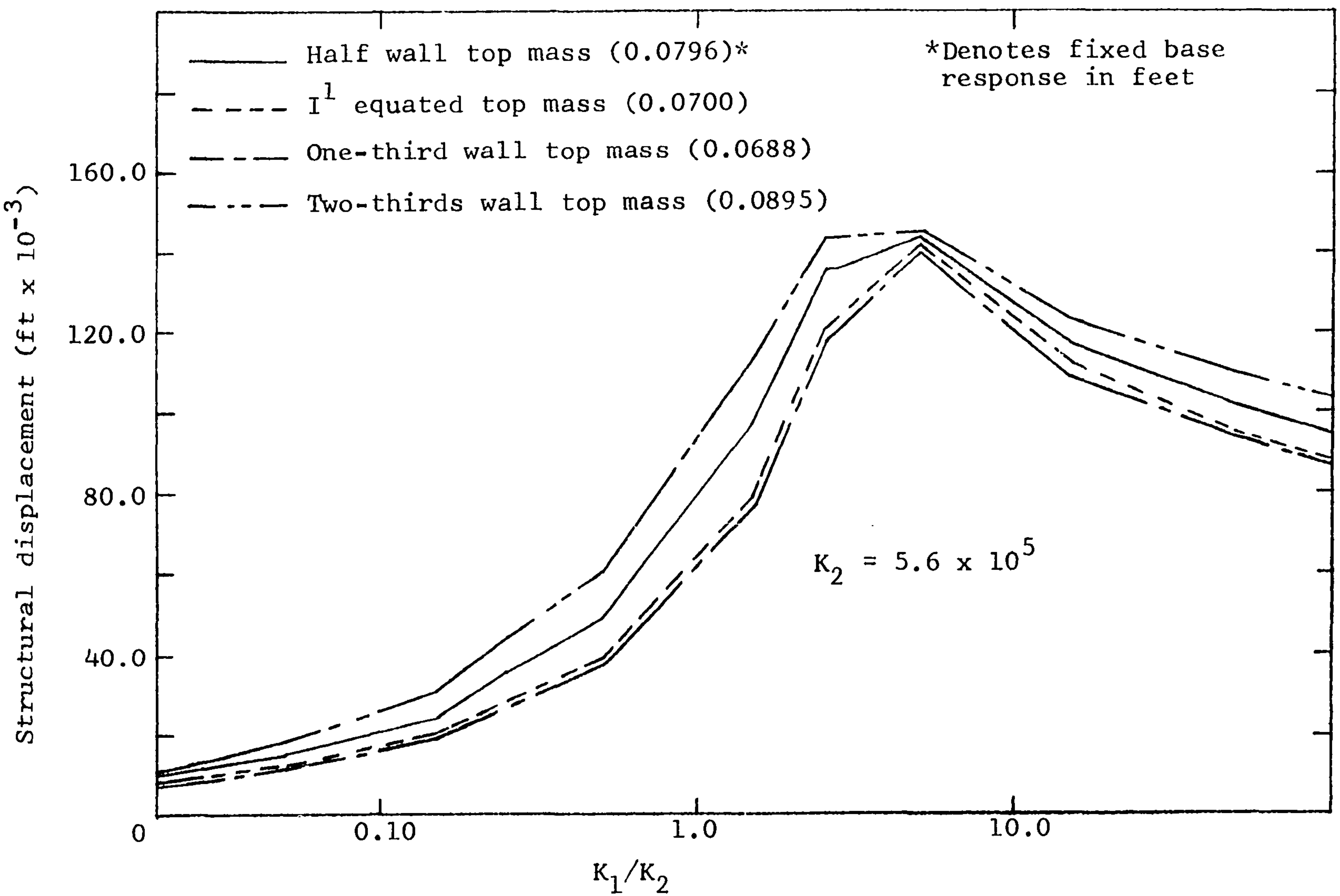


Figure 25. Type II translational structure displacement as a function of stiffness ratio

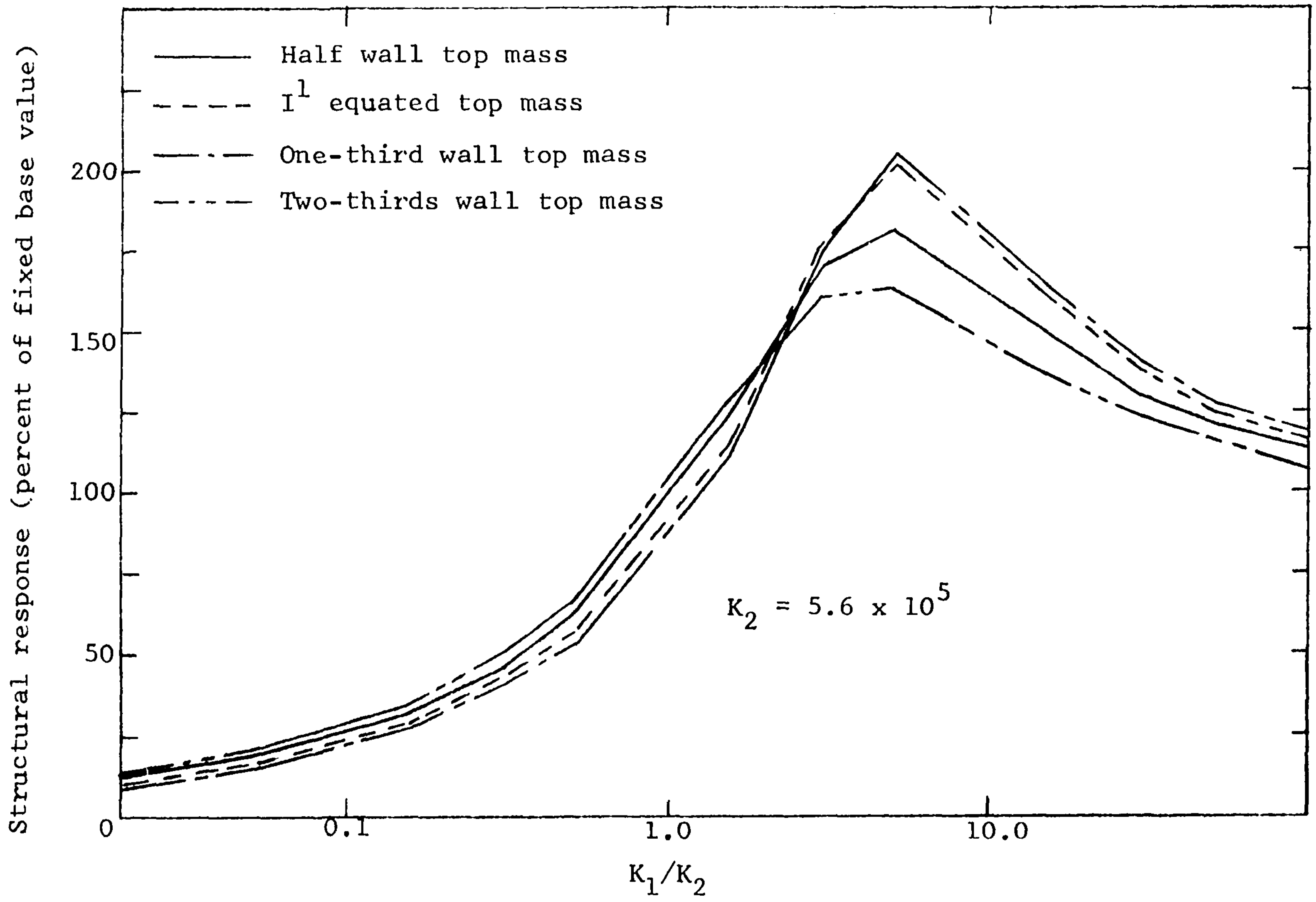


Figure 26. Type II translational structure response as a function of stiffness ratio

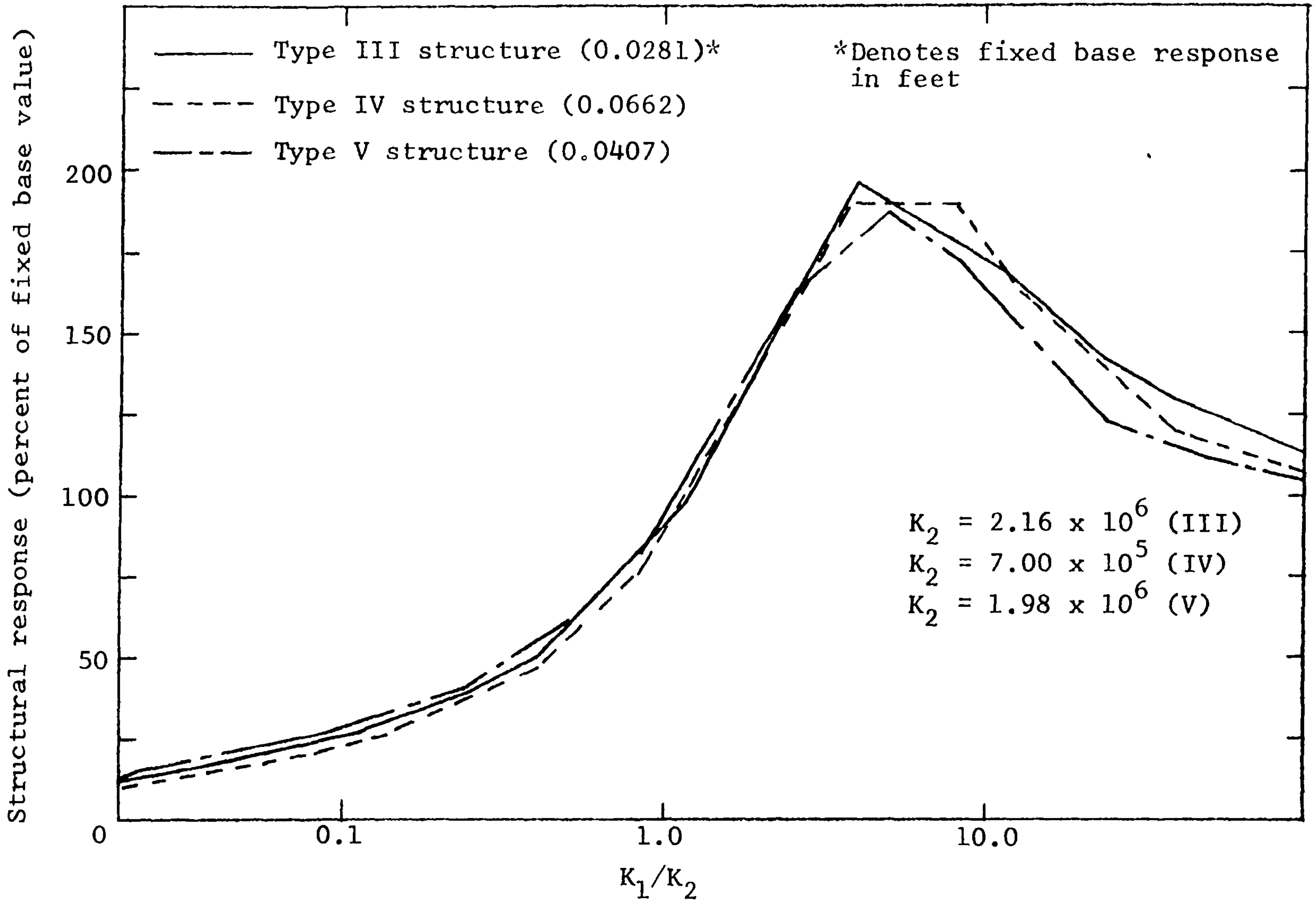


Figure 27. Type III, IV and V translational structure response as a function of stiffness ratio (I^1 equated top mass)

structures with rotational type foundations is shown in figures twenty-eight through thirty-two. Specifically, in figure twenty-eight the maximum structural displacement developed in a rotationally based type I structure when subjected to an El Centro type ground motion is graphed as a function of the stiffness ratio (K_r/K_2). The four curves in the figure again show the results obtained when the four different mass partitions are used. In figure twenty-nine essentially the same information, plotted in terms of structural response rather than structural displacement, is shown.

Figure thirty and thirty-one illustrate the effect of ground motion on a taller rotationally based structure, the rotational type II structure. Here again displacement versus stiffness ratio is graphed in figure thirty and structural response versus stiffness ratio is graphed in figure thirty-one. In figure thirty-two the response of type III, IV and V rotational structures to the same El Centro ground motion is shown for the I^1 equated mass partition.

4. Structures with rotational and translational freedom

Proceeding on to the physical situation where the foundation possesses both translational and rotational freedom (called the combined structure) the situation becomes somewhat more complex in as much as one is now dealing with a three degree-of-freedom system with many different rotational/translational combinations possible. This problem can, in reality, be viewed as a function surface whose Z coordinate is structural displace-

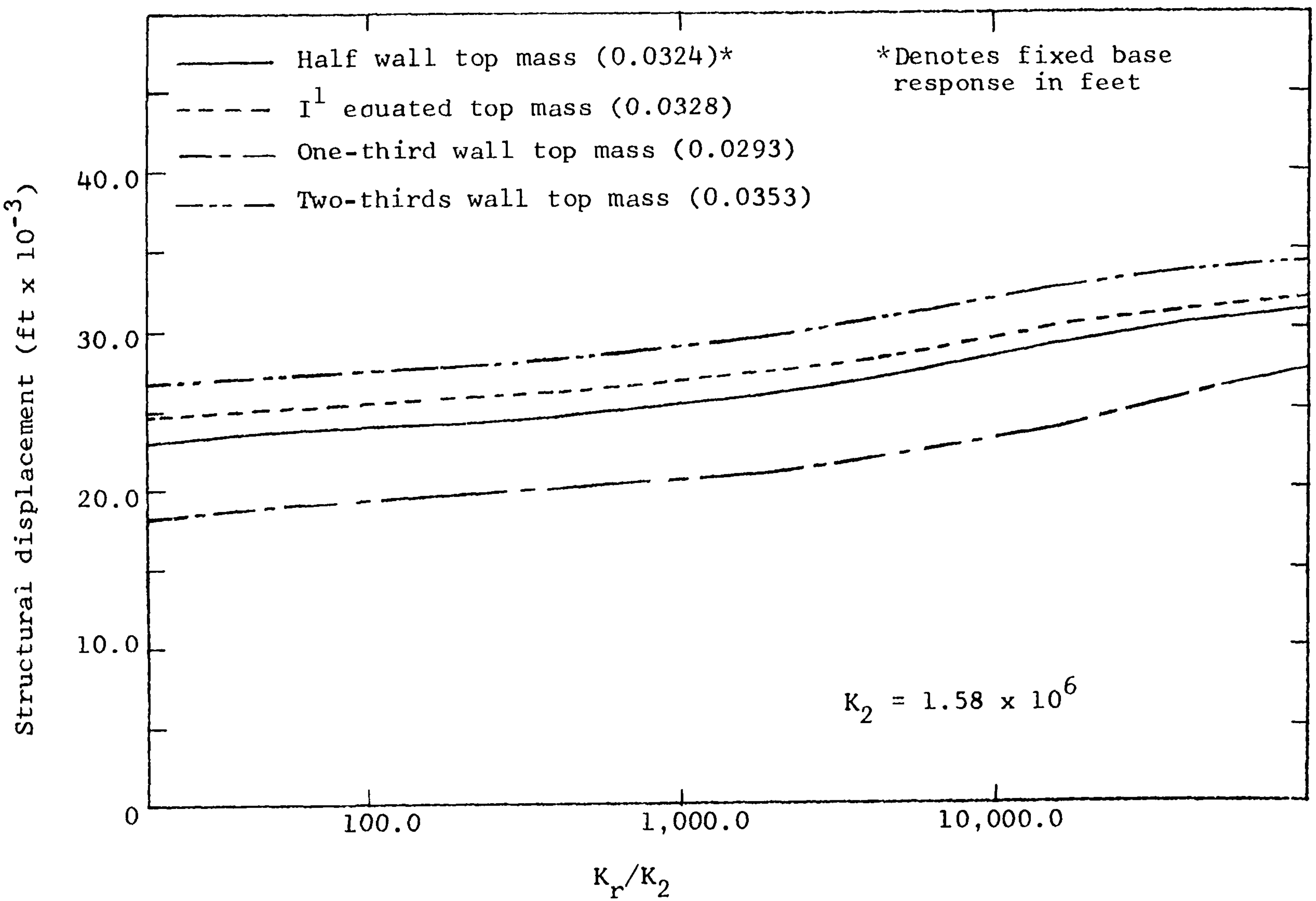


Figure 28. Type I rotational structure displacement as a function of stiffness ratio

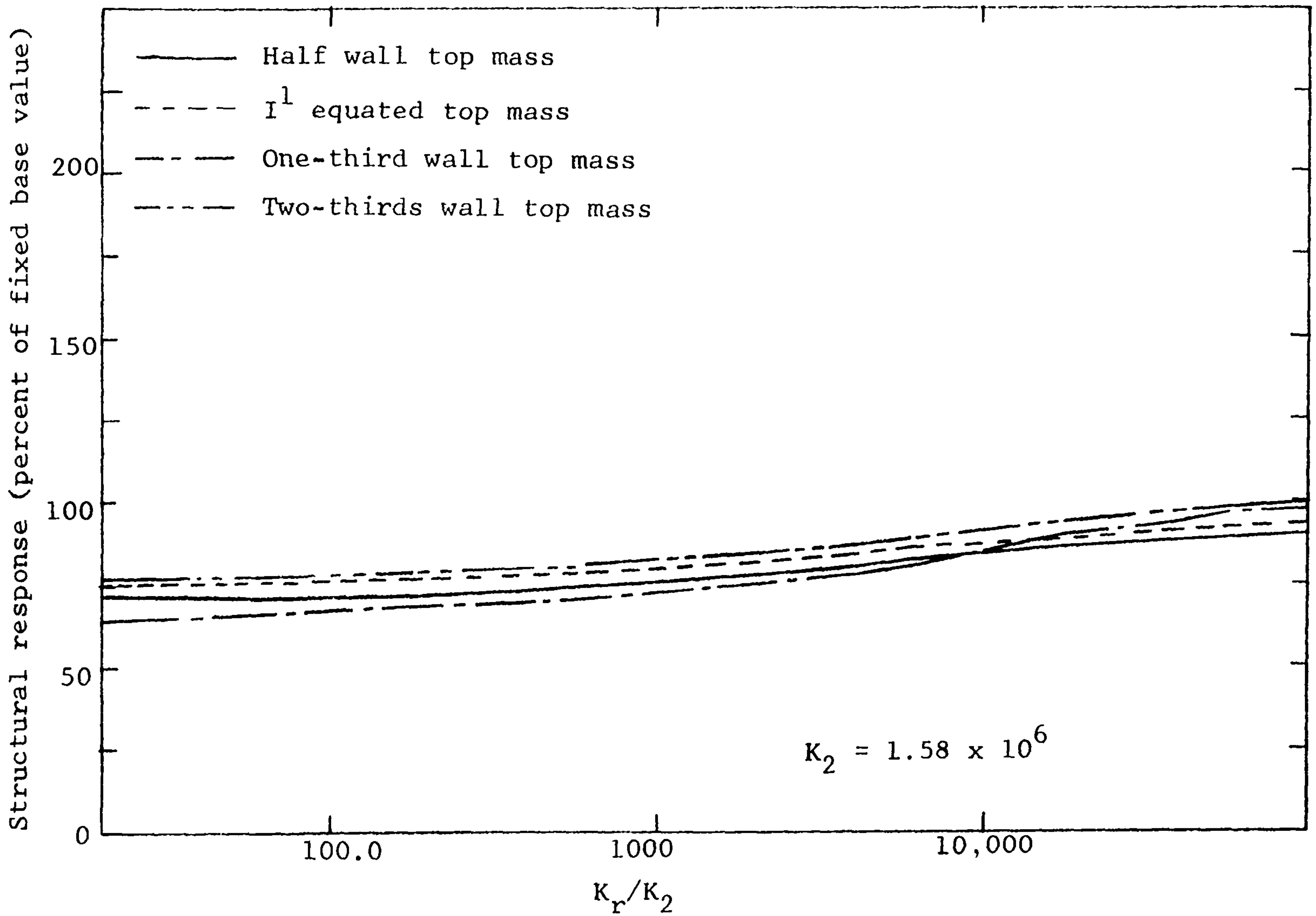


Figure 29. Type I rotational structure response as a function of stiffness ratio

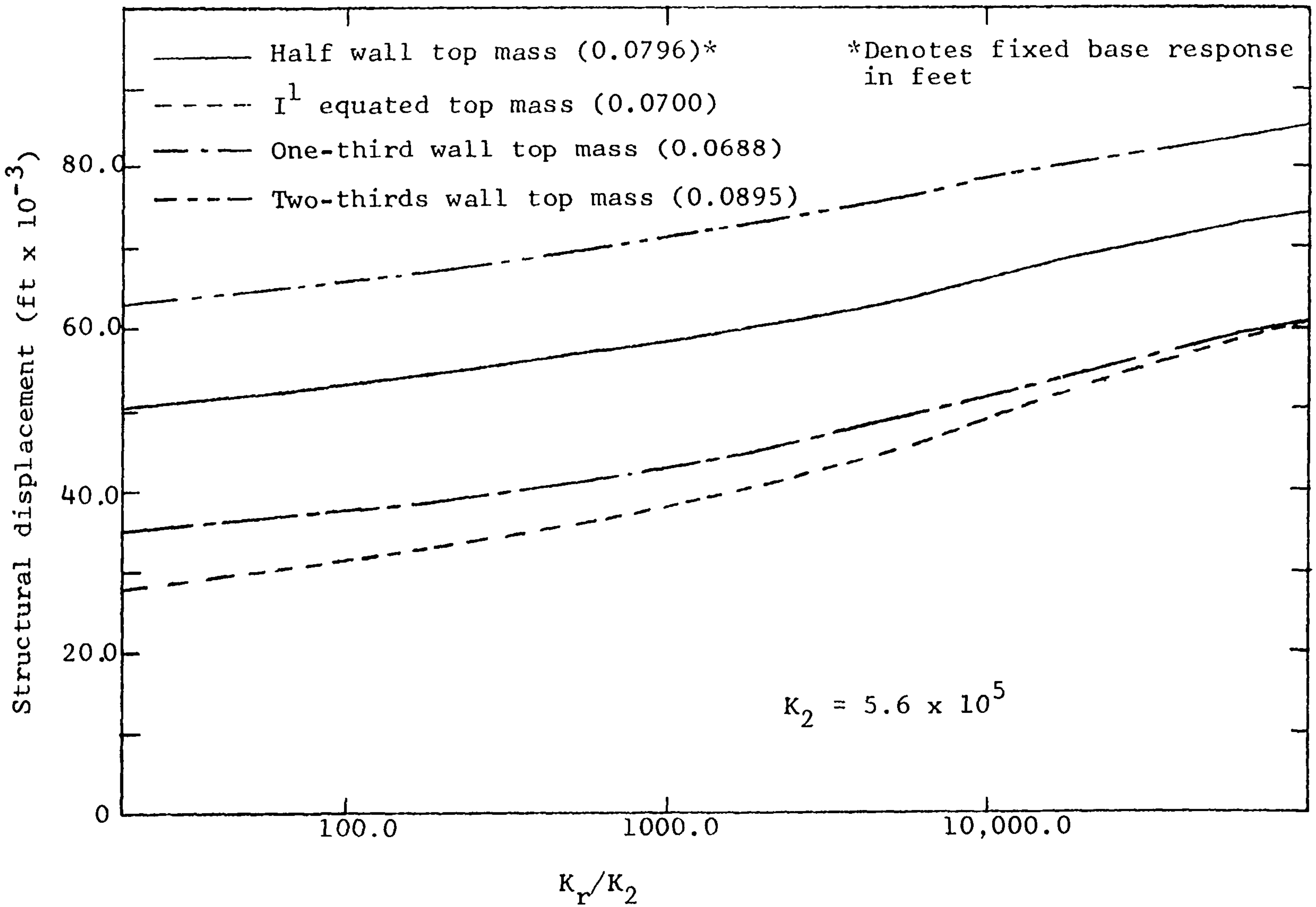


Figure 30. Type II rotational structure displacement as a function of stiffness ratio

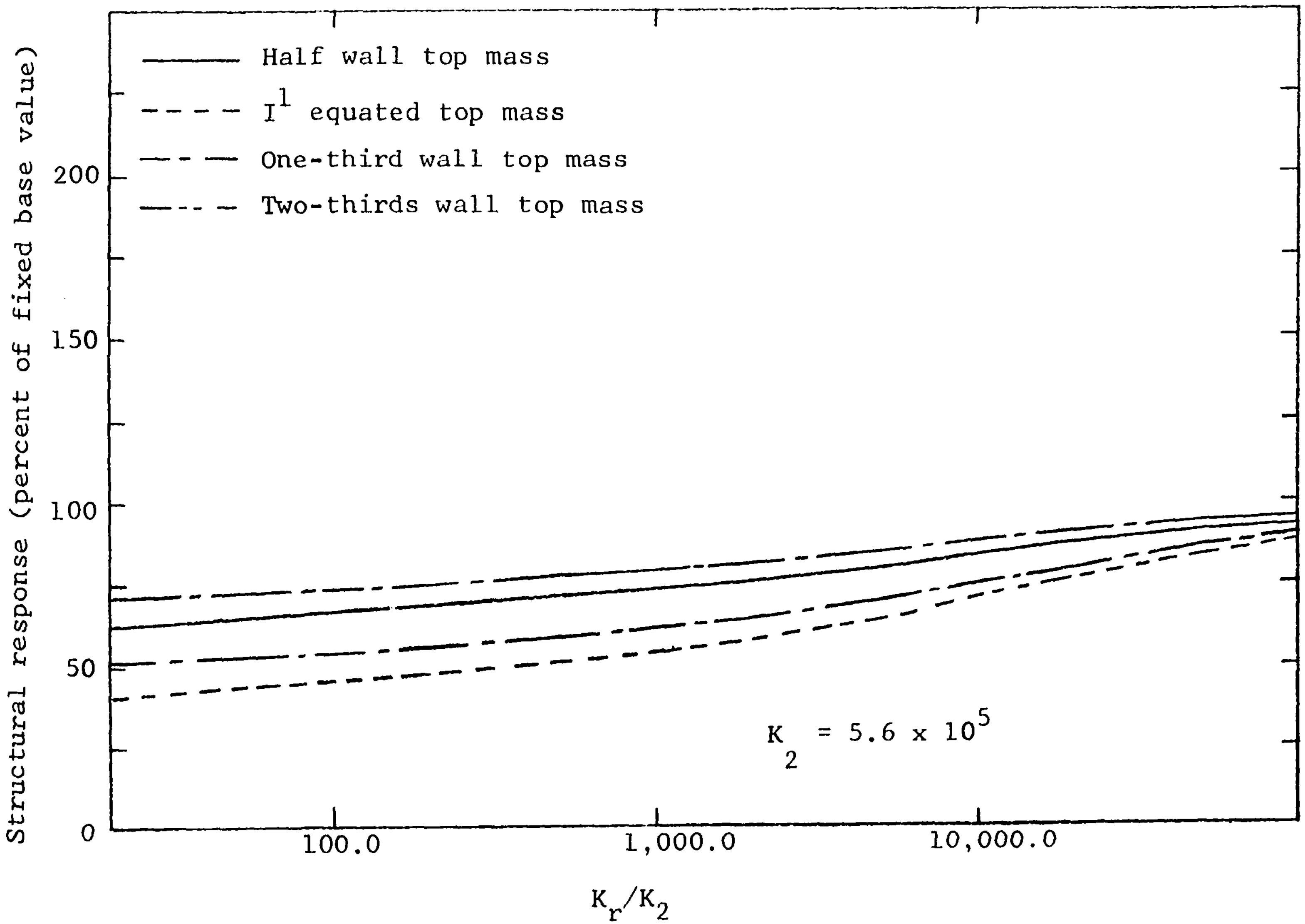


Figure 31. Type II rotational structure response as a function of stiffness ratio

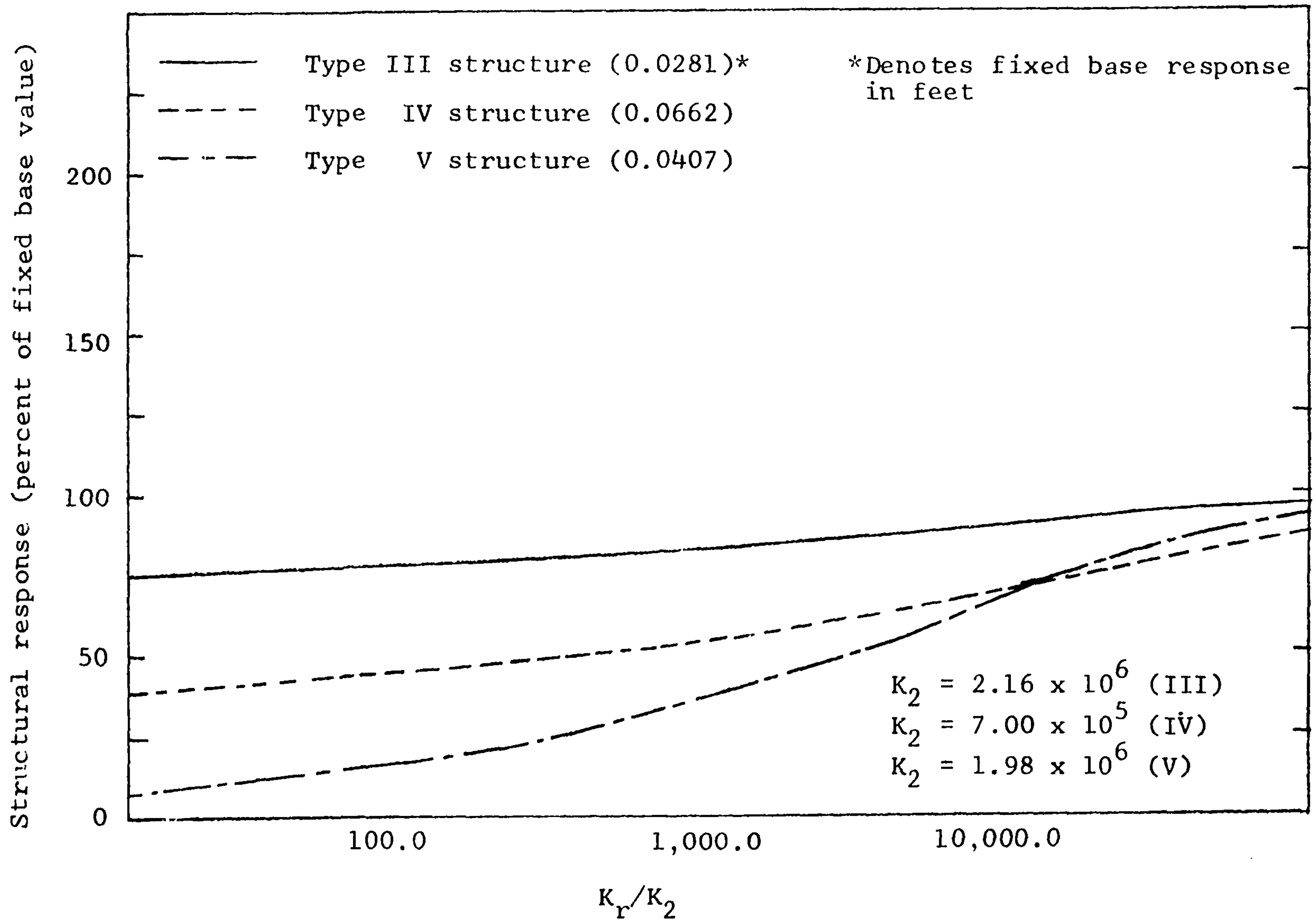


Figure 32. Type III, IV and V rotational structure response as a function of stiffness ratio (I^1 equated top mass)

ment or structural response and whose X and Y coordinates are the translational and rocking stiffnesses of the specific foundation of concern.

One such surface has been calculated completely using a coarse mesh and the El Centro ground motion for a type I structure with I^1 equated mass partitioning. Two views of this surface are shown in figures thirty-three and thirty-four. The function surface for the type II, III, IV and V structures all have the same general shape of surface as this one shown for the type I structure.

The function surface has been examined in more detail for each containment type for two subcases that are considered to have special relevance. They are the cases a) where the soil system foundation is such that a general proportional relationship exists between soil translational and rotational stiffnesses and b) where the rocking stiffness is essentially constant but the translational stiffness may vary over a wide range. The first subcase is thought typical of a normal soil foundation. The second subcase is thought to apply to a typical vertical pile foundation where the rocking stiffness is a known constant value based perhaps on pile load tests and/or the elasticity of the piles themselves. In this second case the translational stiffness may vary over a wide range depending on soil type, pile type, pile spacing, pile depth and pile batter.

The containment structure with a foundation with both translational and rotational freedom and a soil foundation of the proportional type is termed here after in the text and figures as combined variable K_r structure. The results of the computations involving this foundation type are shown in figures thirty-five through thirty-nine. The ratio between foundation rotational stiffness and foundation translational stiffness has been held constant at a value of 3550 for type I, II, III and IV structure and 6150 for the type V structure. The selection of these particular values is based on an average relationship between compressive and shear stiffness reported in the literature (1) and consideration of the foundation areas involved.

The maximum relative displacement occurring in a type I combined variable K_r structure under El Centro type ground motion is graphed in figure thirty-five as a function of stiffness ratio. This same information is, also, presented in figure thirty-six. However, in this figure the response ratio is graphed as a function of stiffness ratio over the range of interest. In figures thirty-seven and thirty-eight the same type information is given for the taller type II structure. Again note that for these four figures various mass partitions have been used.

The influence of ground motion on the structural response for a range of foundation stiffnesses for the combined variable K_r type III, IV and V structures has also been calculated. The results of these calculations for the I^1 equated mass parti-

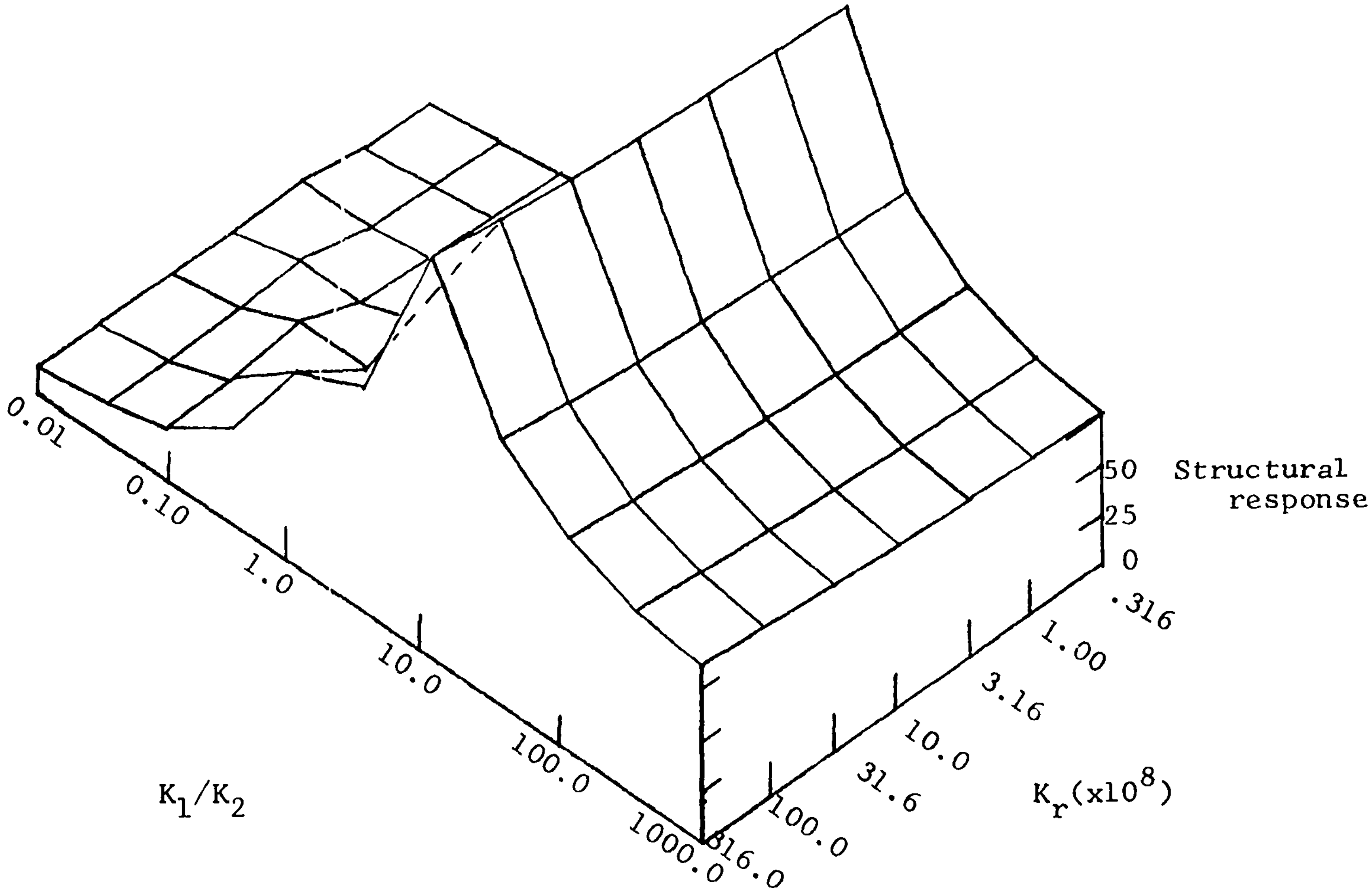


Figure 33. Function surface for type I structure

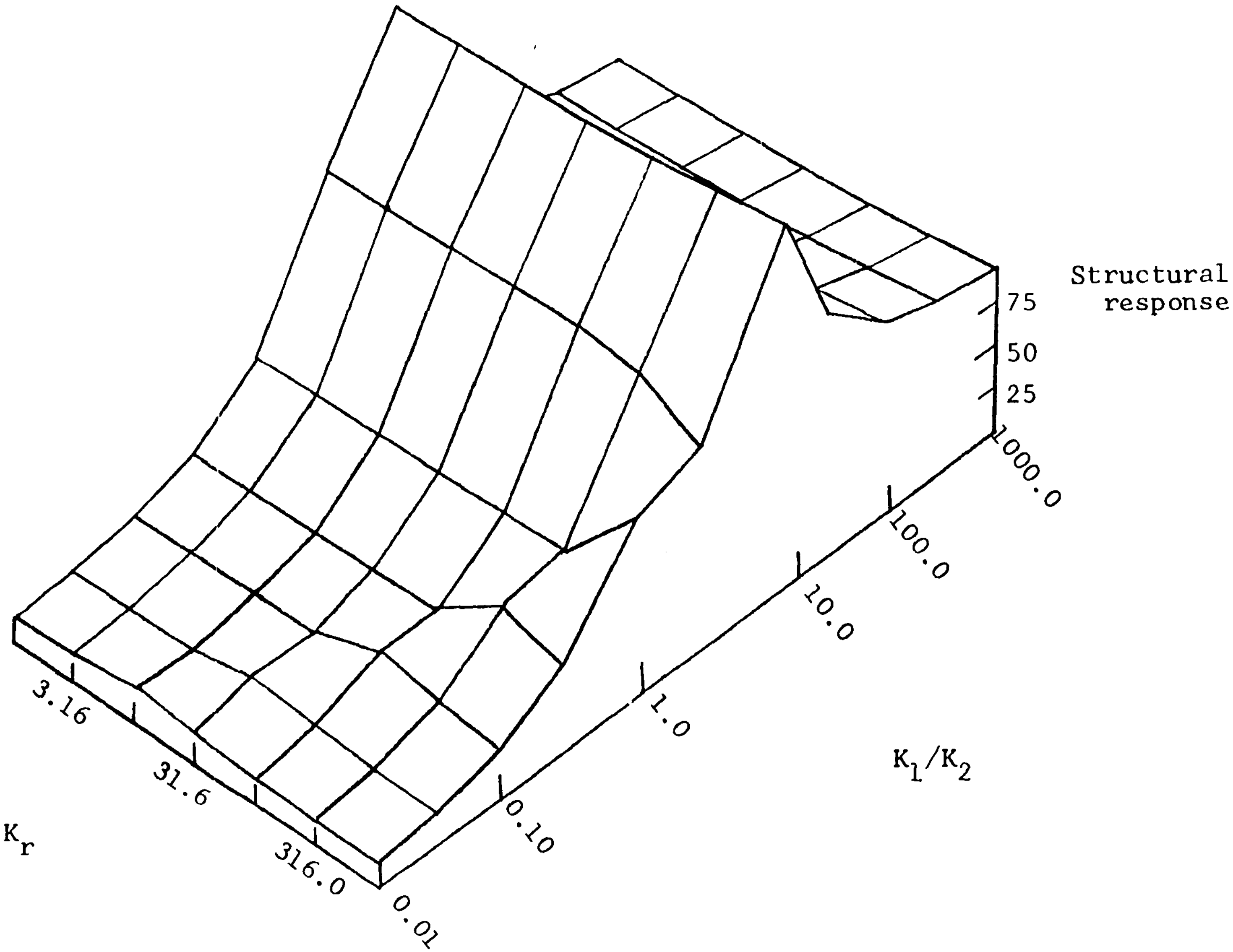


Figure 34. Function surface for type I structure

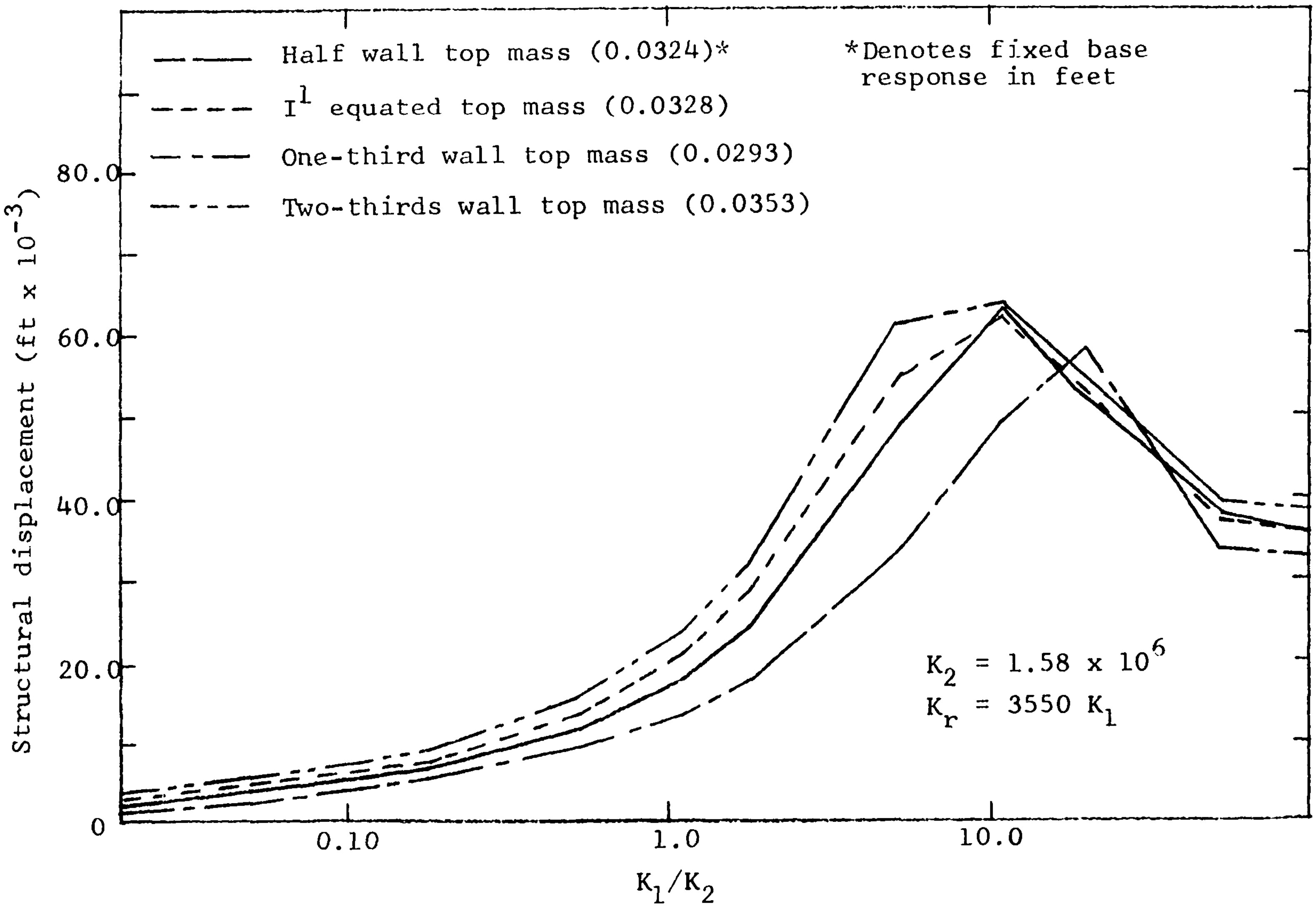


Figure 35. Type I combined (variable K_r) structure displacement as a function of stiffness ratio

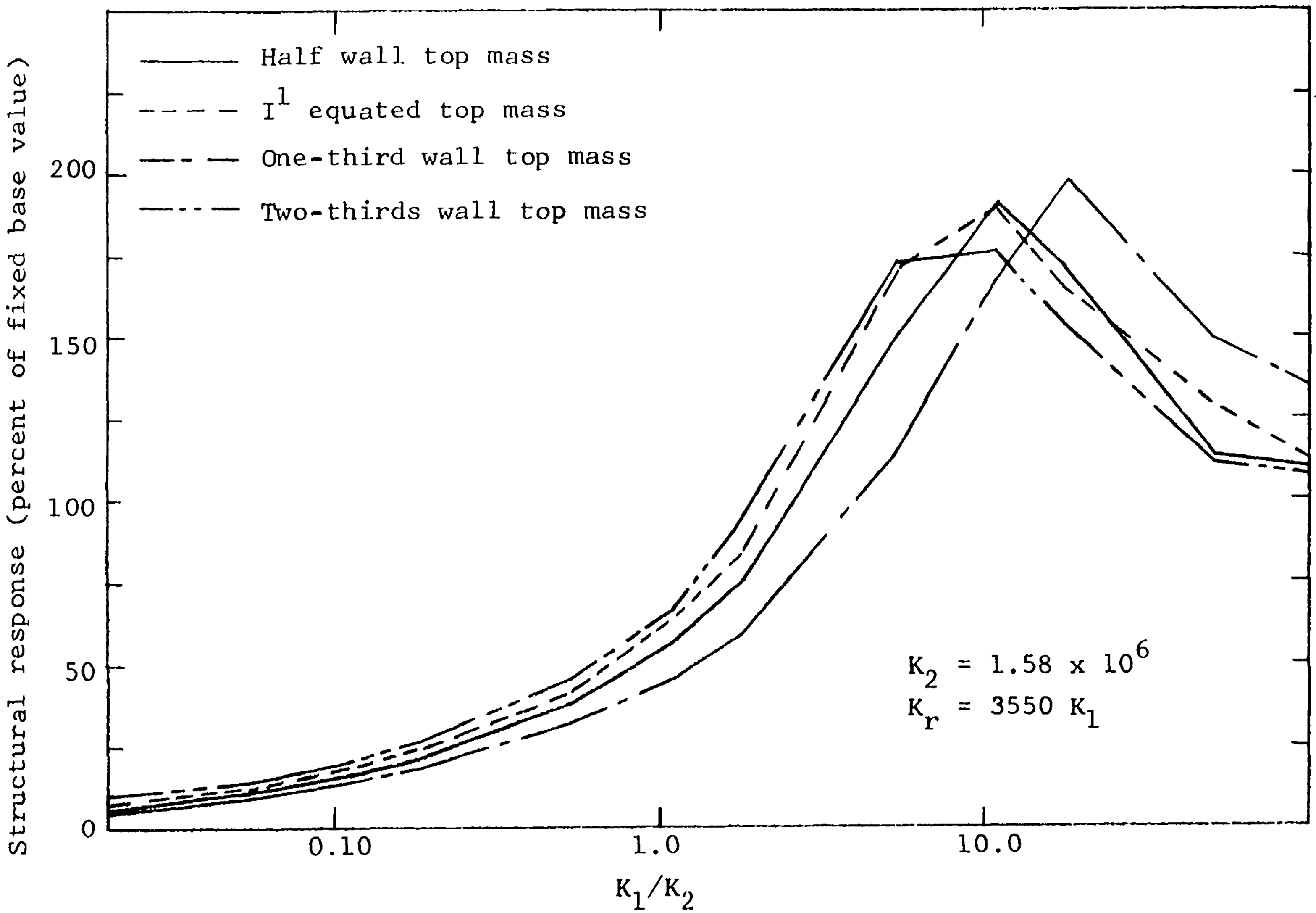


Figure 36. Type I combined (variable K_r) structure response as a function of stiffness ratio

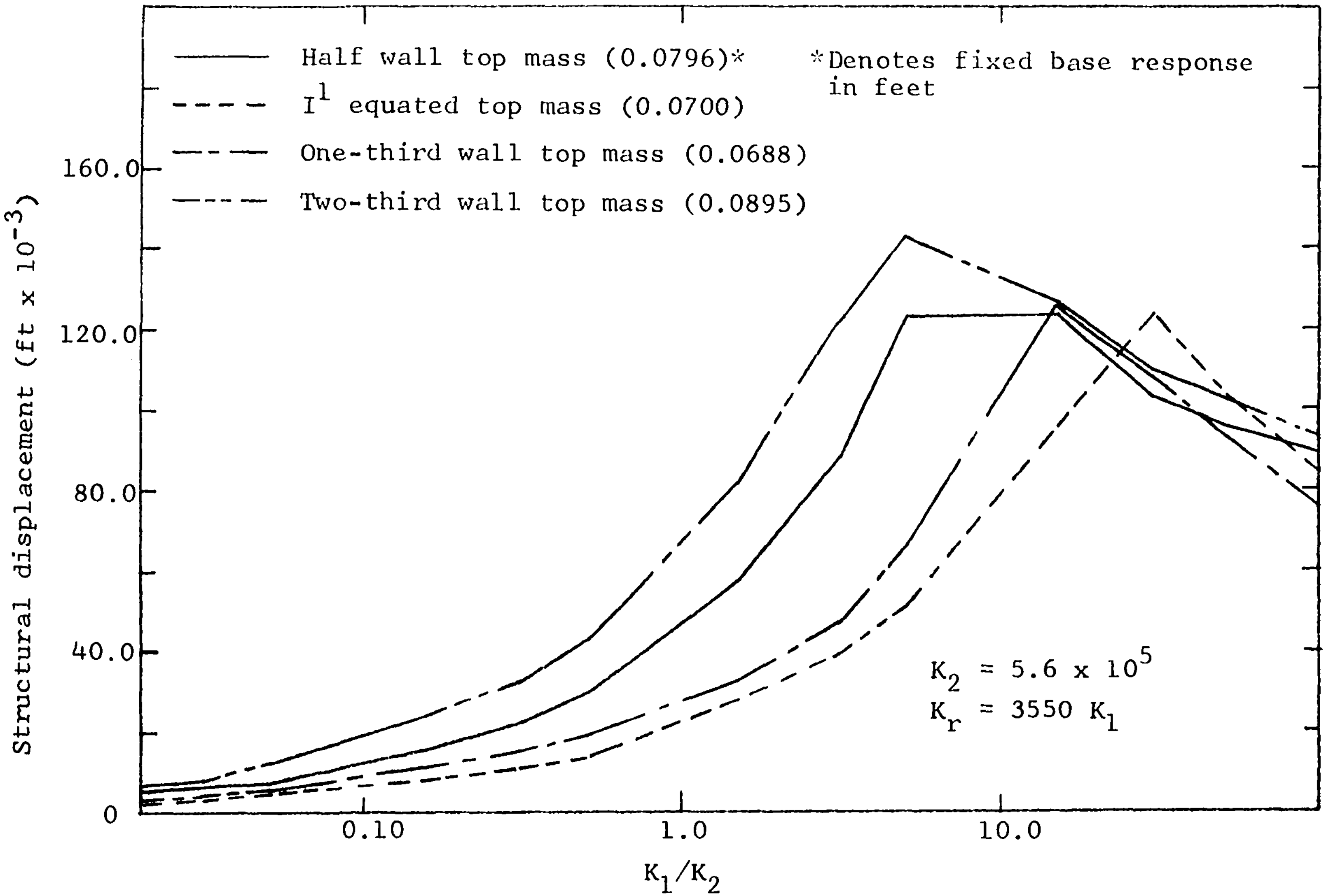


Figure 37. Type II combined (variable K_r) structure displacement as a function of stiffness ratio

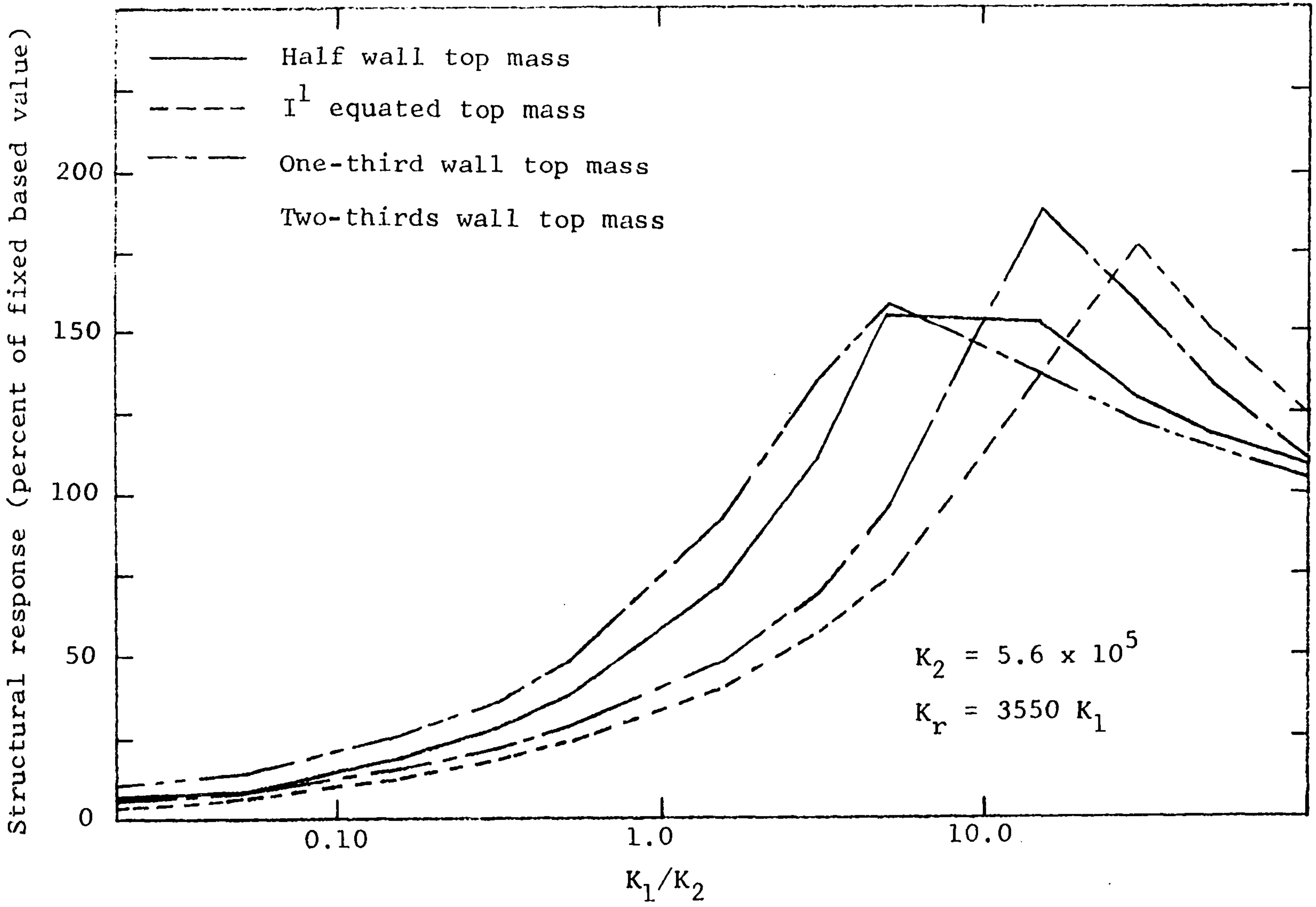


Figure 38. Type II combined (variable K_r) structure response as a function of stiffness ratio

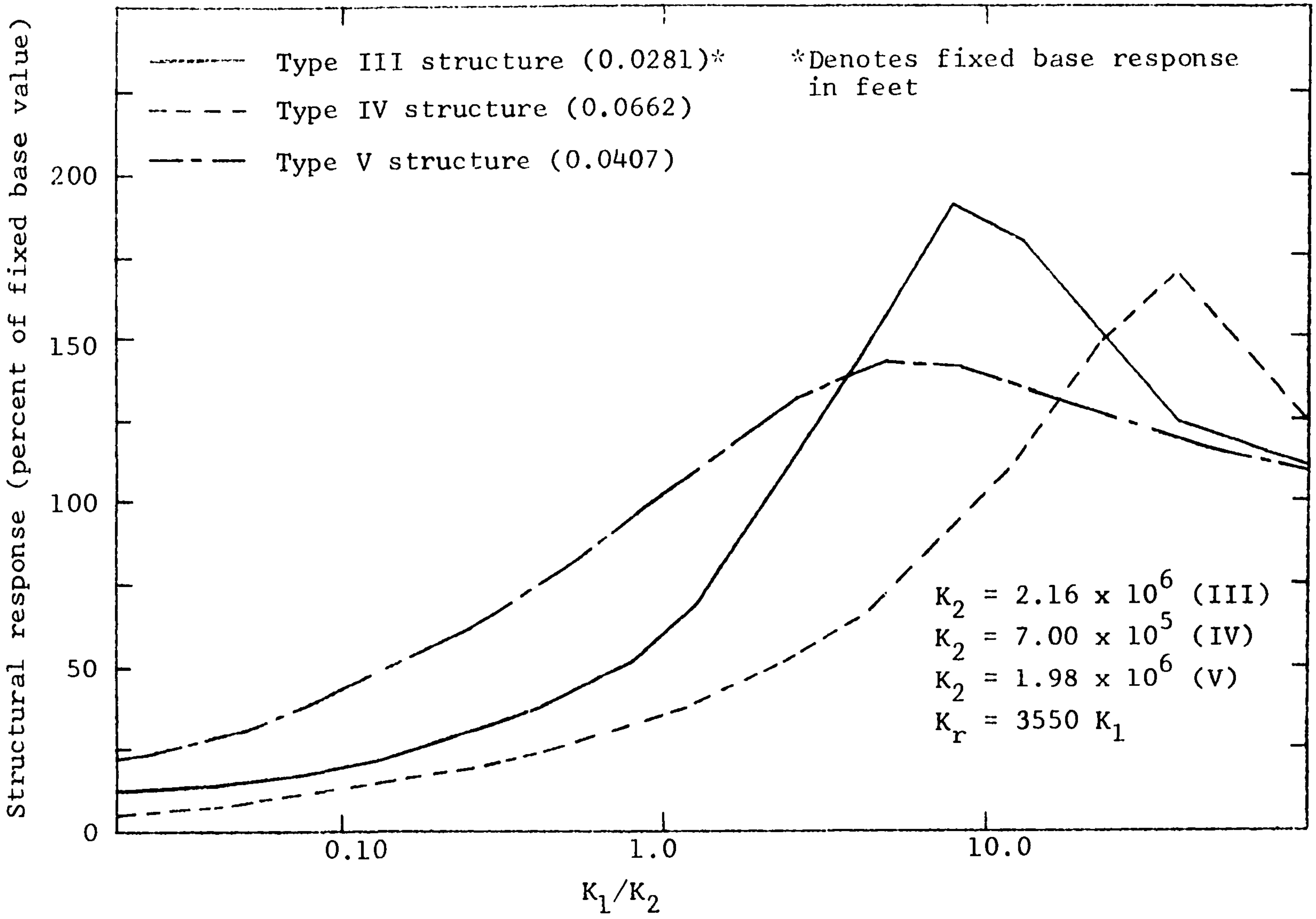


Figure 39. Type III, IV and V combined (variable K_r) structure response as a function of stiffness ratio

tioning has also been calculated. The results of these calculations are graphed in figure forty-four.

5. Omission of mass and moment of inertia

In the previous calculations the base mass (M_1) included the mass of equipment and structures internal to the containment as well as a portion of the containment structure, itself. In like manner, the moment of inertia used in the calculations was the total moment of inertia of all masses, both internal and containment structure, about the base of the containment.

The effect of omission of the mass and moment of inertia of internal equipment is of interest. The containment designer must many times complete the design before internal structures and equipment are accurately specified as to location and weight. It is, therefore, desirable to determine the effect on the dynamic response of the containment of neglect of the masses and moments of inertia of items which are normally attached to the containment's foundation mat. Figures forty-five through forty-eight present the results of structural response calculations for a type II containment structure with various foundation freedoms where equipment and internal structure masses and/or moment of inertia are not included in the mathematical model. In each figure the structural response for the same structure with internal mass and moment of inertia included has also been graphed to provide a reference for comparison.

Figure forty-five shows the effect of omission of mass and/

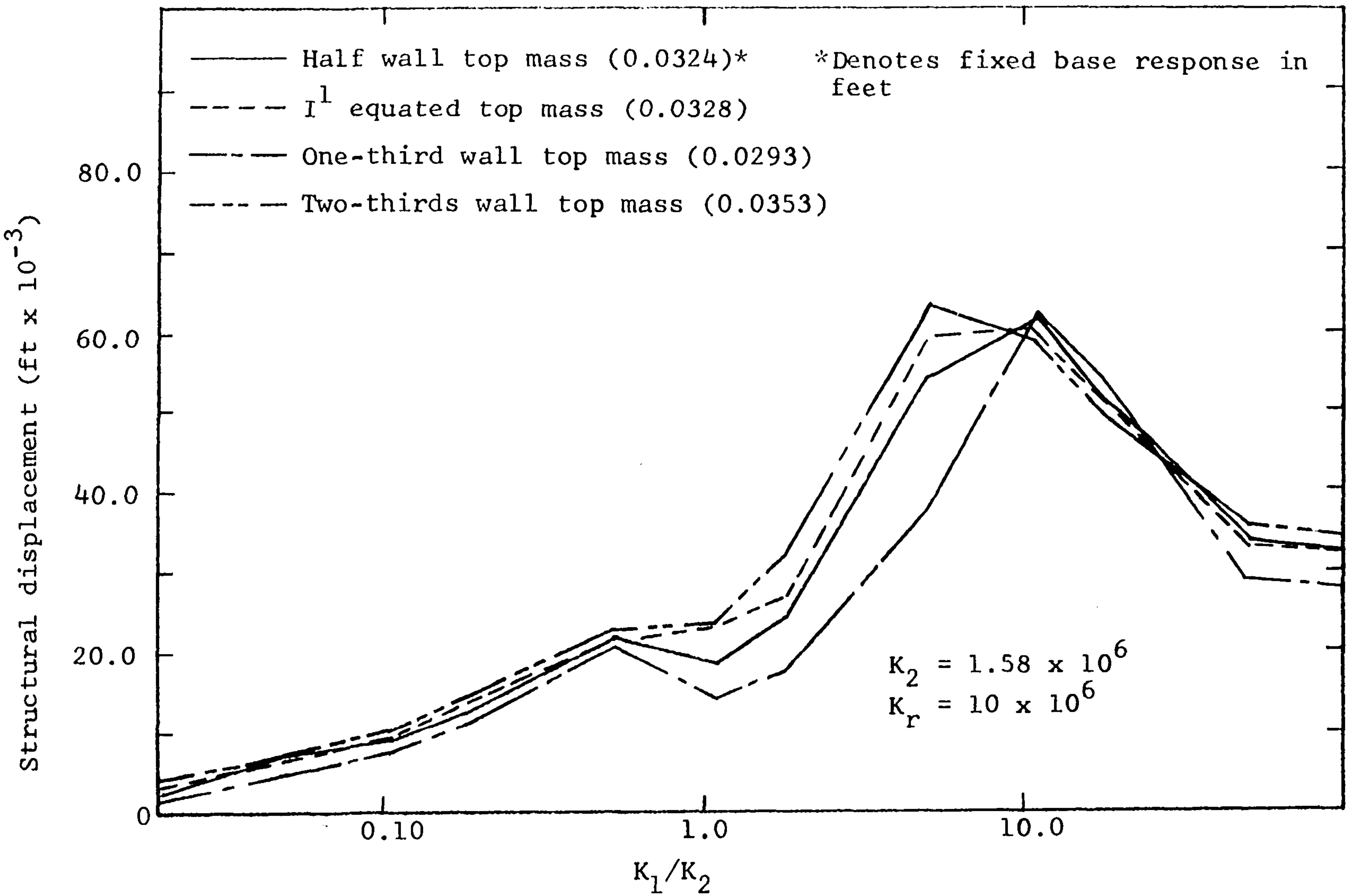


Figure 40. Type I combined (constant K_r) structure displacement as a function of stiffness ratio

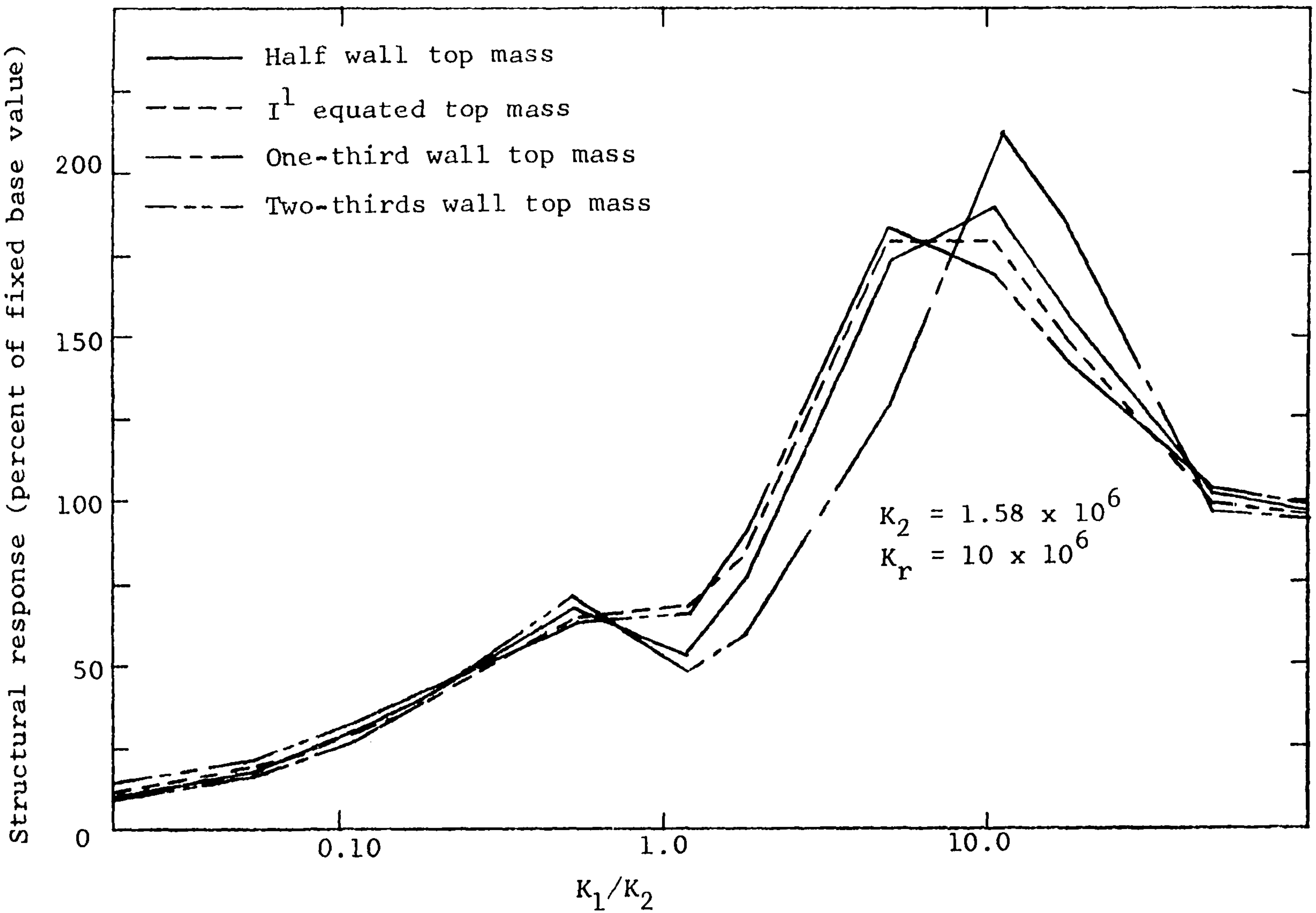


Figure 41. Type I combined (constant K_r) structure response as a function of stiffness ratio

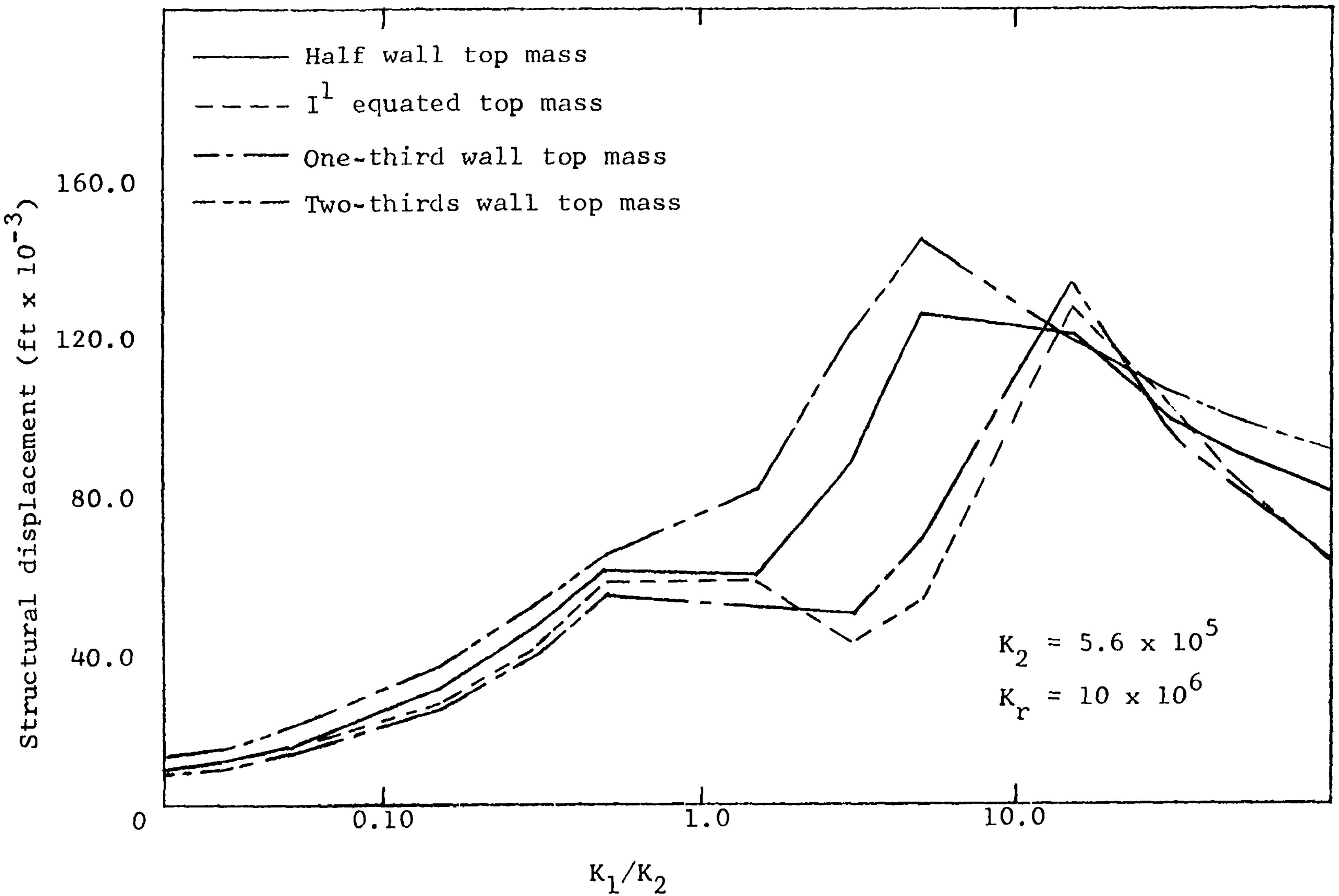


Figure 42. Type II combined (constant K_r) structure displacement as a function of stiffness ratio

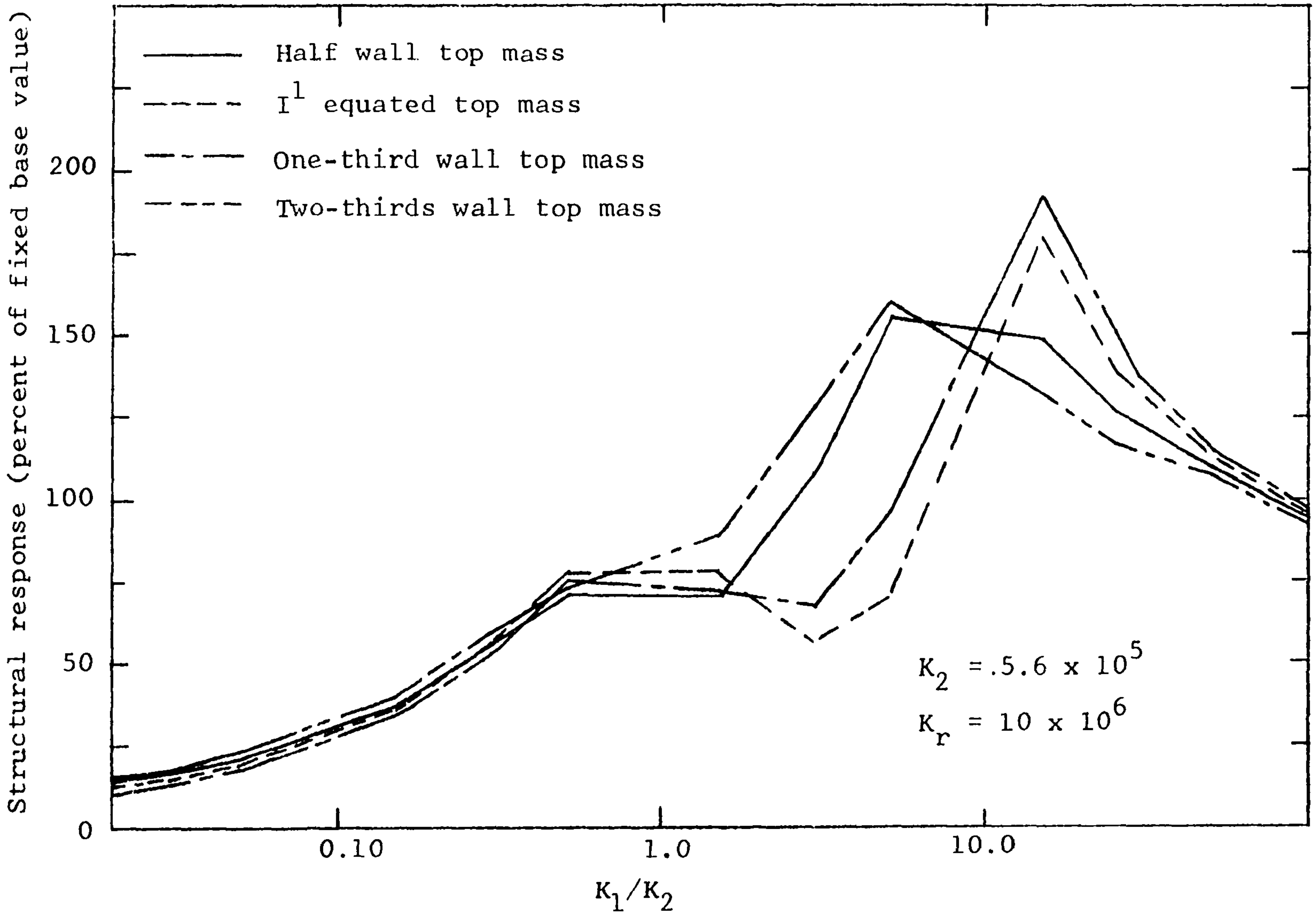


Figure 43. Type II combined (constant K_r) structure response as a function of stiffness ratio

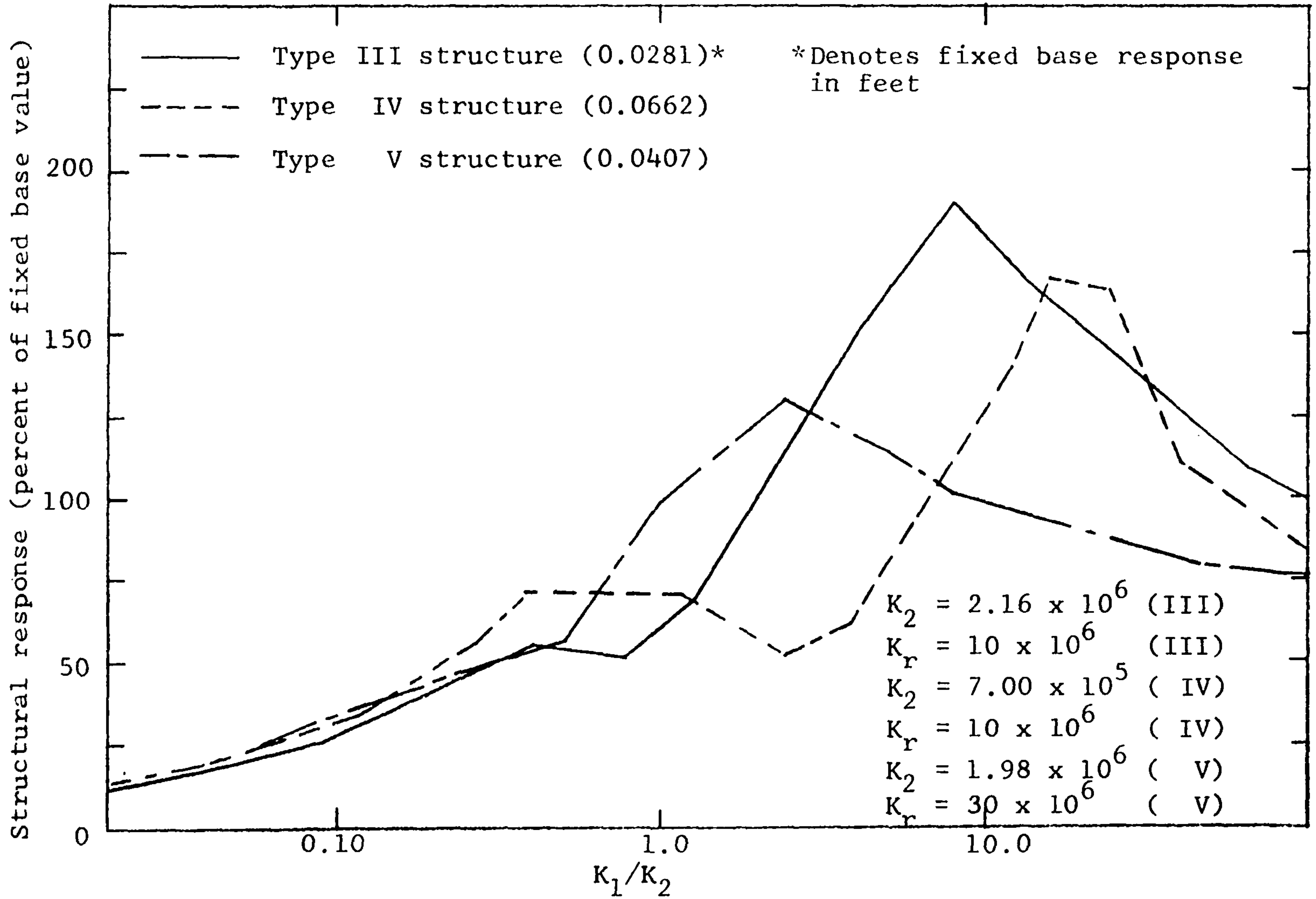


Figure 44. Type III, IV and V combined (constant K_r) structure as a function of stiffness ratio

or moment of inertia on the response of a type II structure with a translational foundation. The structural response is graphed, as previously, as a function of foundation stiffness ratio. Figures forty-six, forty-seven, and forty-eight show the result of omission of mass and/or moment of inertia on the structural response of rotational, combined variable K_r and combined constant K_r structures, respectively.

D. Analog and Digital Undamped Structural Response Due to Sinusoidal Loading

The damping phase of this investigation was accomplished through the use of an analog computer technique. To support the validity of the patched analog setup to be used in subsequent damping studies, the analog program with zero damping was run with the selected sinusoidal ground motion input. The result, in terms of structural displacement, of this input for type I and II structures, varying foundation situations and varying degrees of foundation stiffness is graphed in figures forty-nine through fifty-four.

For comparative purposes modal analyses were also made for the same structural models and using the same forcing function. The response spectrum for this ground motion forcing function is shown in figure twelve. The structural displacement using sum of modal maxima (upper bound), square root of the sum of the squares of modal maxima and differences between the modal maxima (lower bound) was obtained. The structural displacement results of these analyses are also graphed in

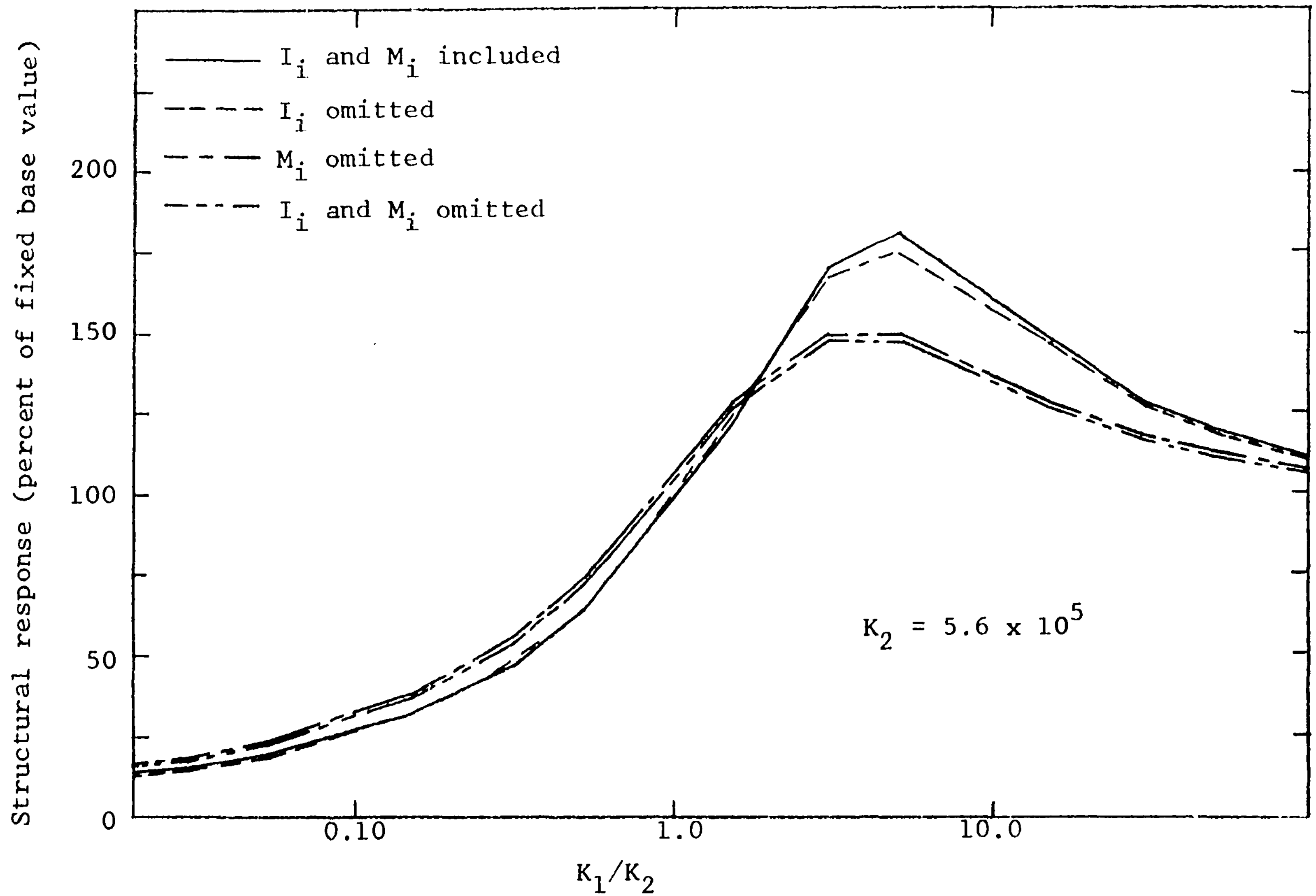


Figure 45. Type II translational structural response as a function of stiffness ratio without considering internal mass and/or moment of inertia

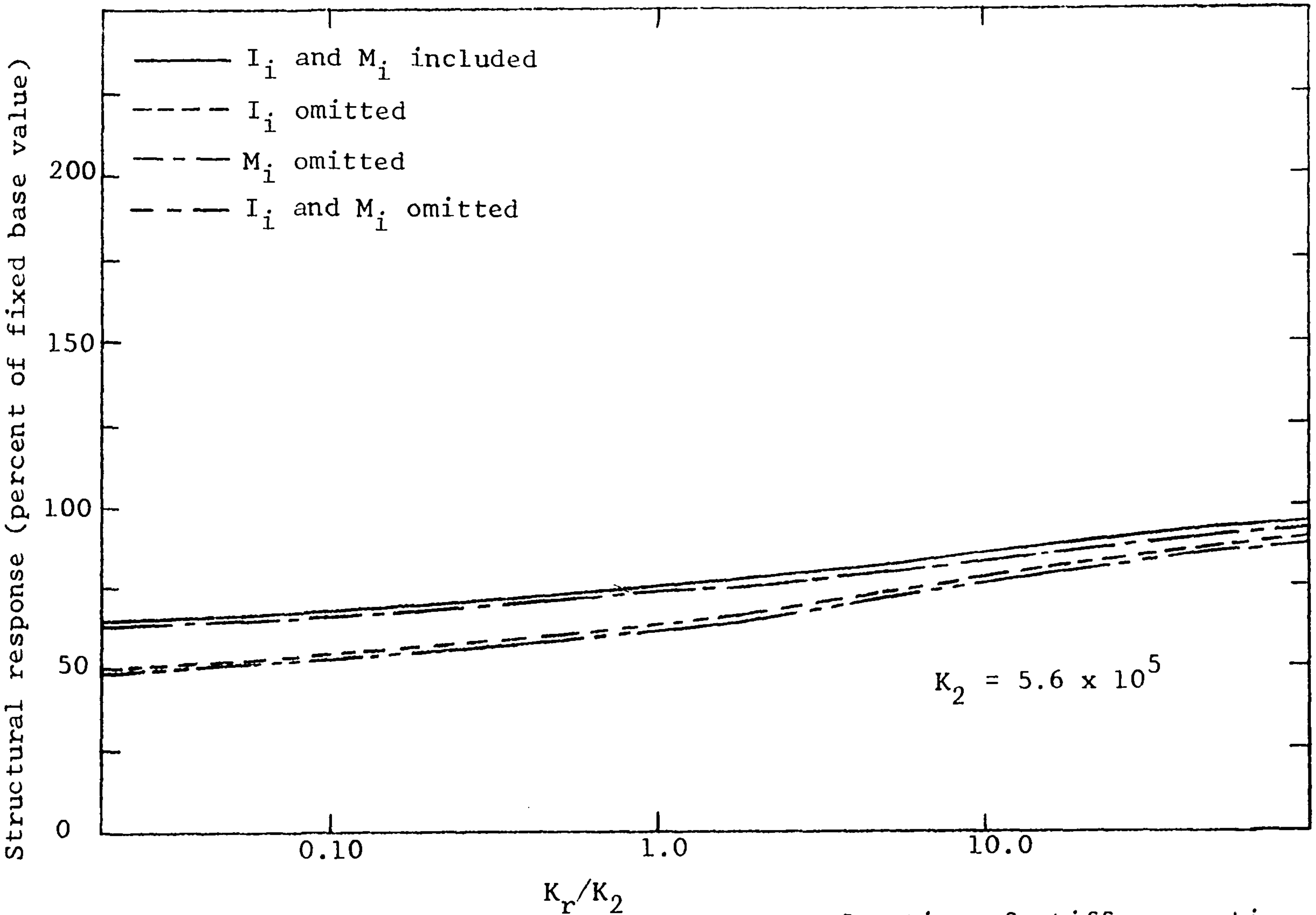


Figure 46. Type II rotational structural response as a function of stiffness ratio without considering internal mass and/or moment of inertia

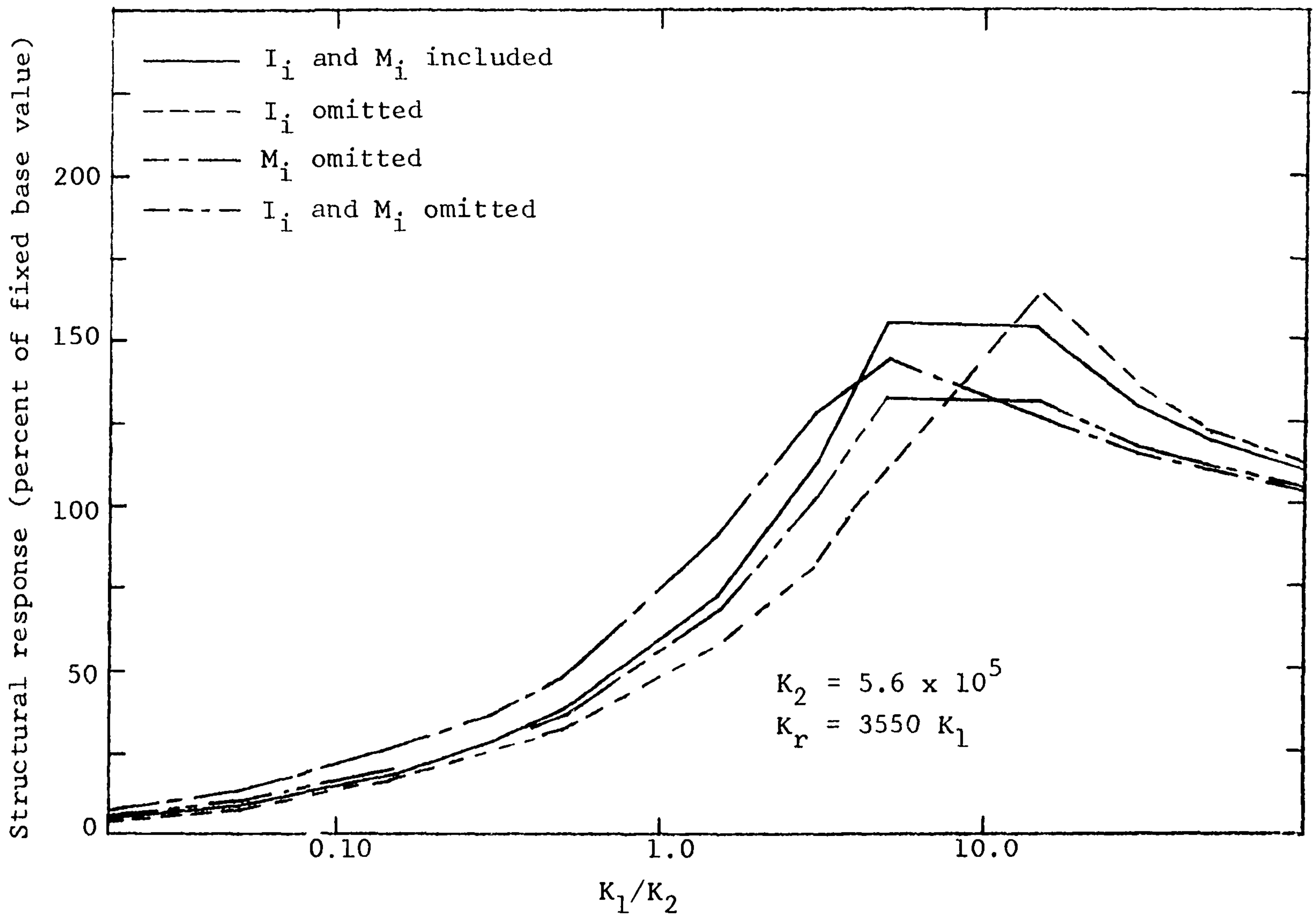


Figure 47. Type II combined variable K_r structural response as a function of stiffness ratio without considering internal mass and/or moment of inertia

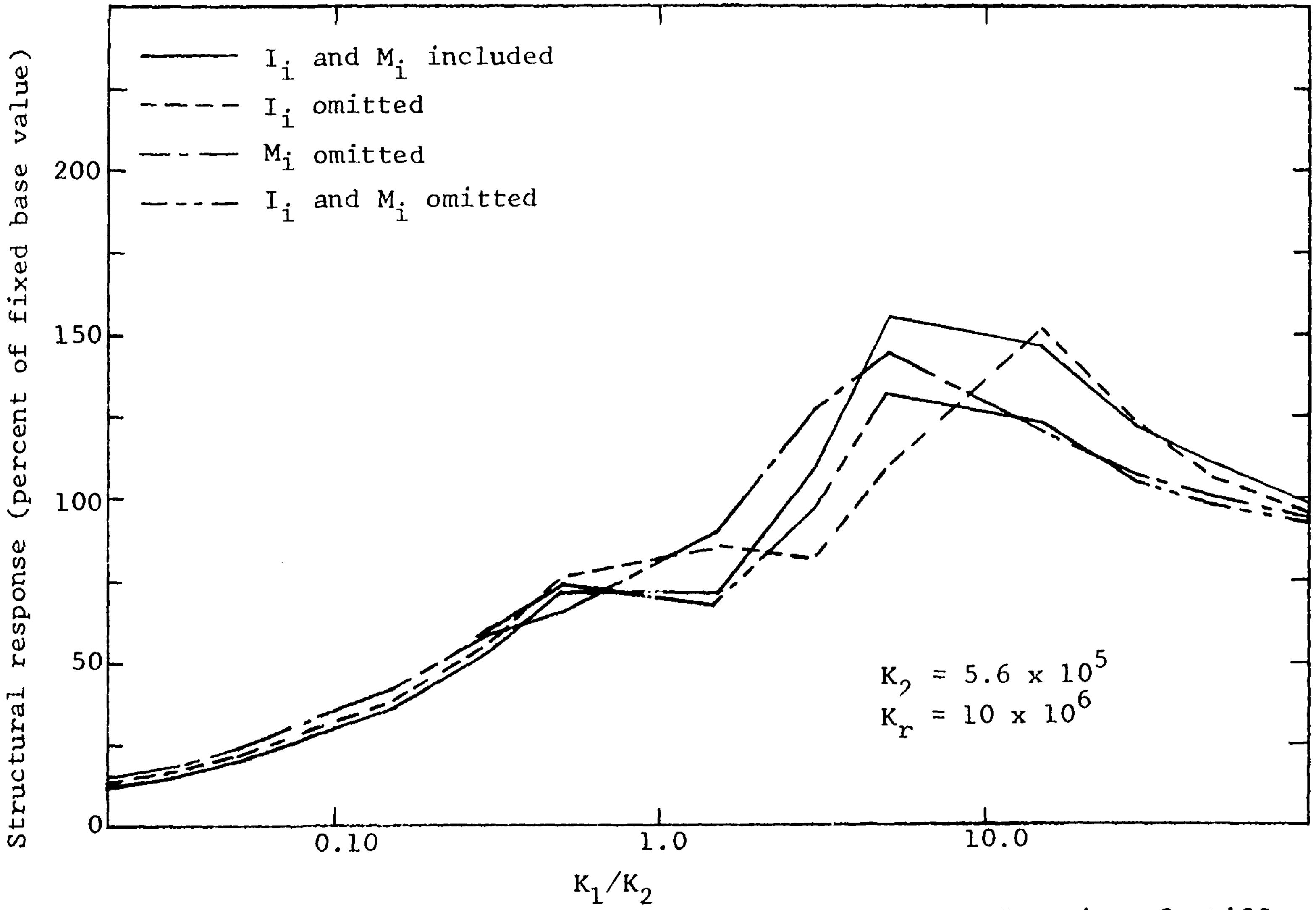


Figure 48. Type II combined constant K_r structural response as a function of stiffness ratio without considering internal mass and/or moment of inertia

figures forty-nine through fifty-four.

More specifically, figures forty-nine, fifty, and fifty-one display the analog and digitally computed structural displacement values for the chosen band of stiffness ratios for type I translational, rotational and combined variable K_r structures, respectively. Structural displacement values were obtained in the same manner for type II translational, rotational and combined variable K_r structures. These values are graphed in figures fifty-two, fifty-three, and fifty-four, respectively.

E. Foundation Damping Influence on Structural Response

As described briefly in the program of investigation section the study of the effect of foundation damping was by analog computer. The type I and type II structures were selected as representative of the range of containment structures. These two types of structures were investigated for translational, rotational and combined variable K_r foundation conditions. For each structure and foundation combination, the foundation stiffness and damping were varied through a wide range. The maximum structural displacement for each of these cases due to the chosen sinusoidal application of ground acceleration was developed. The results of these investigations are shown in figures fifty-five through sixty. For each case investigated the structural damping percentage was held to two percent for comparative purposes.

The damping effect of various percentages of foundation

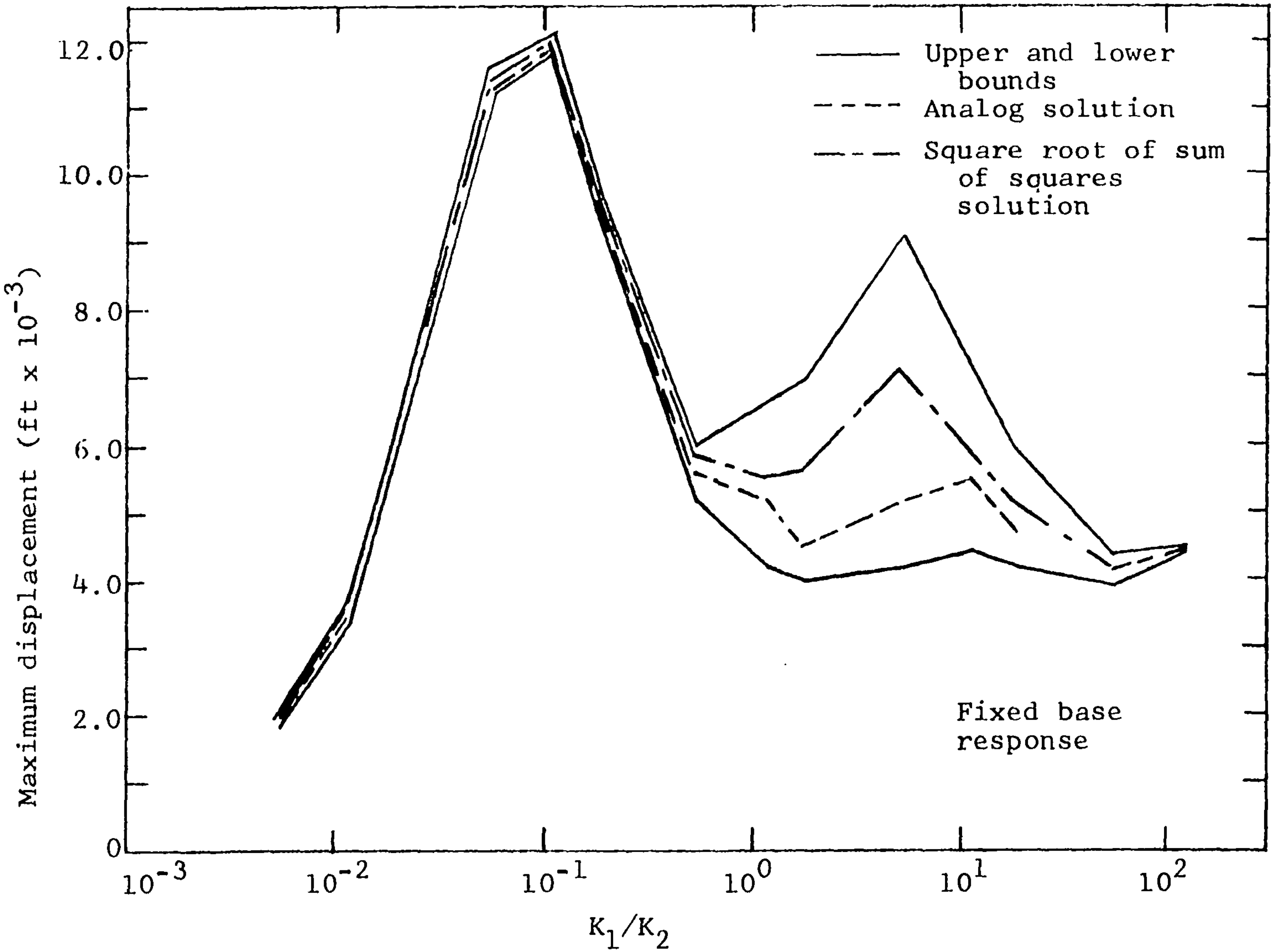


Figure 49. Type I translational structure comparison of analog maximum displacement with that of various modal computed displacement maxima

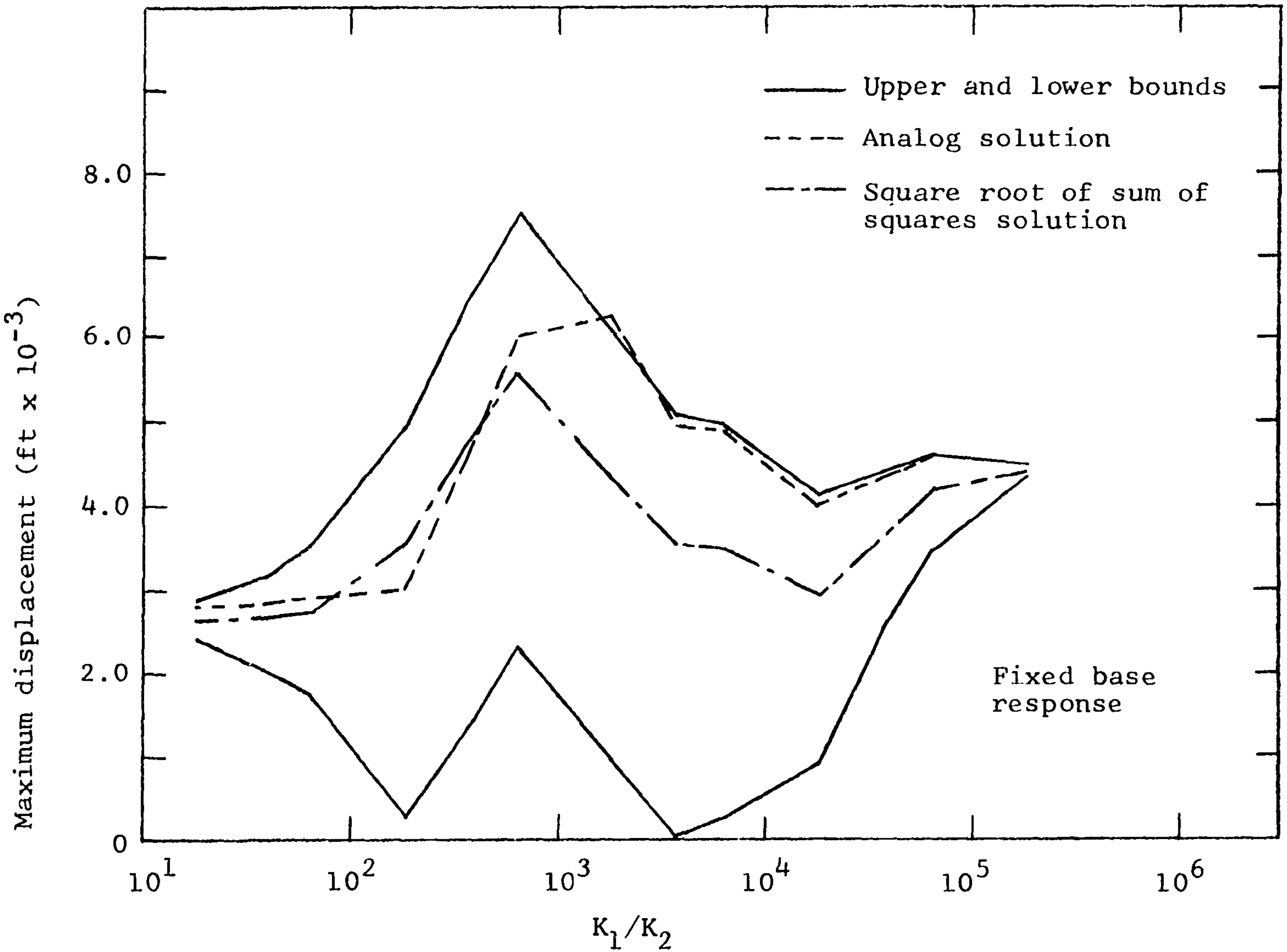


Figure 50. Type I rotational structure comparison of analog maximum displacement with that of various modal computed displacement maxima

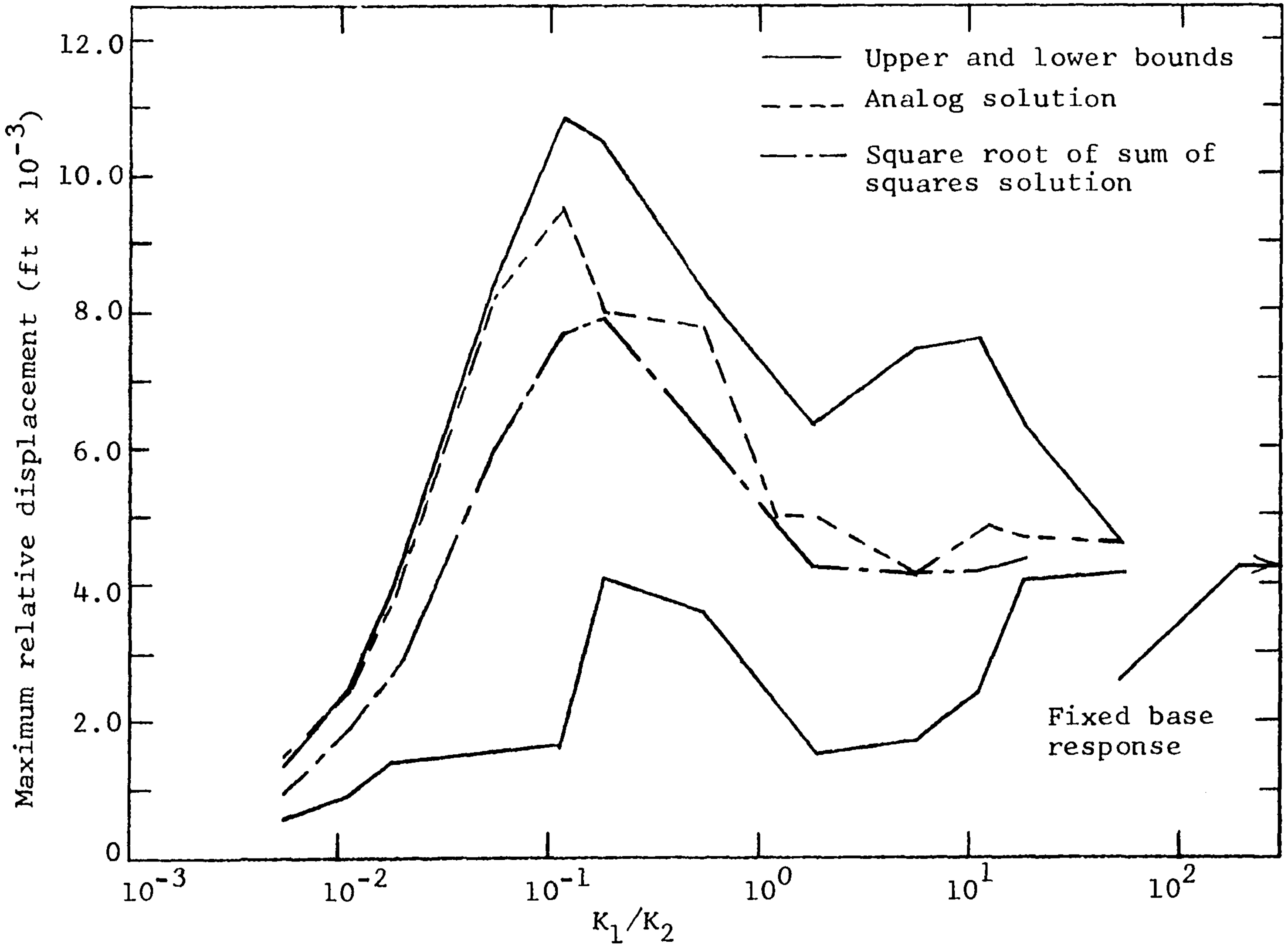


Figure 51. Type I combined variable K_r structure comparison of analog maximum displacement with that of various modal displacement maxima

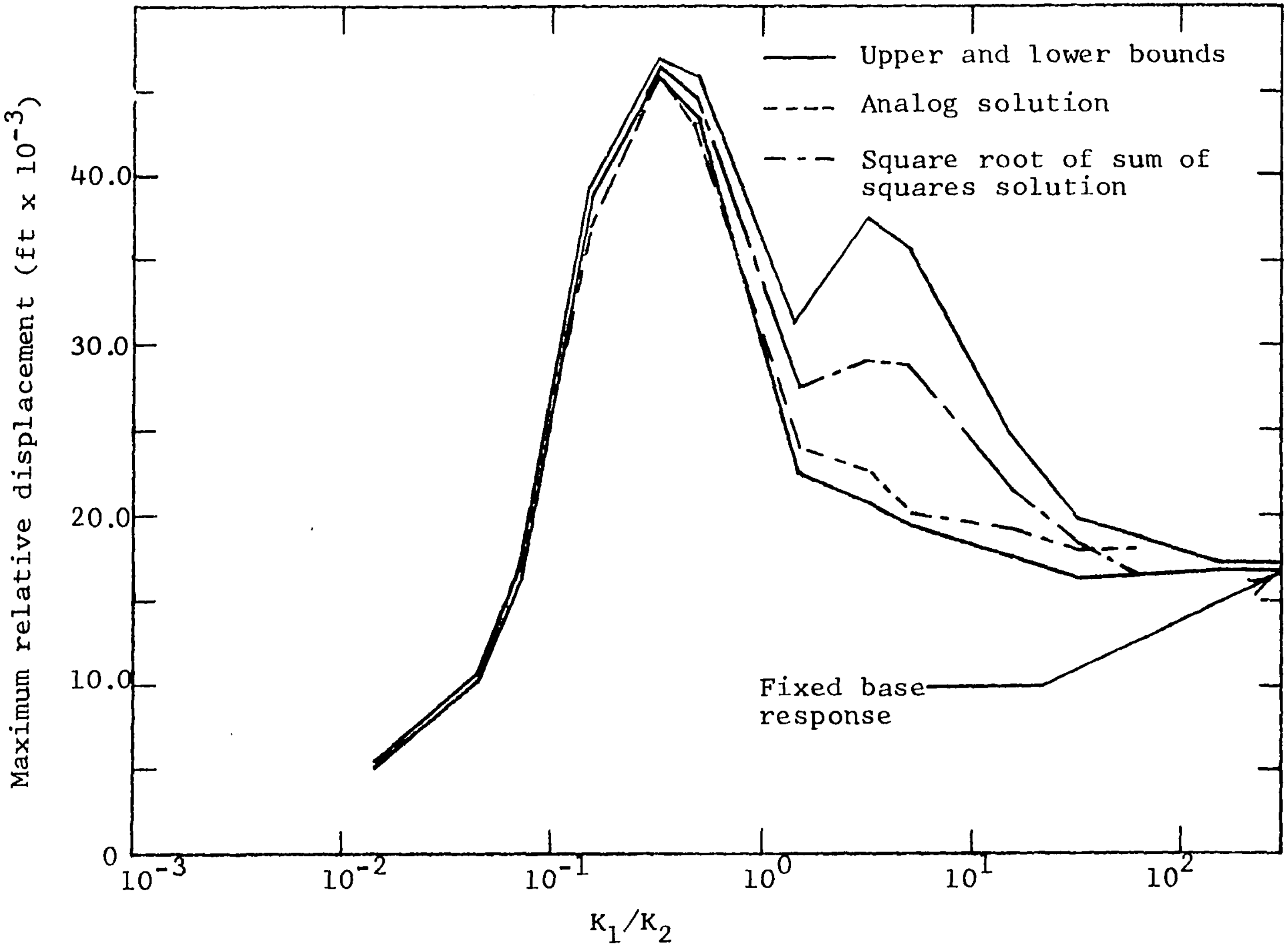


Figure 52. Type II translational structure comparison of analog maximum displacement with that of various modal computed displacement maxima

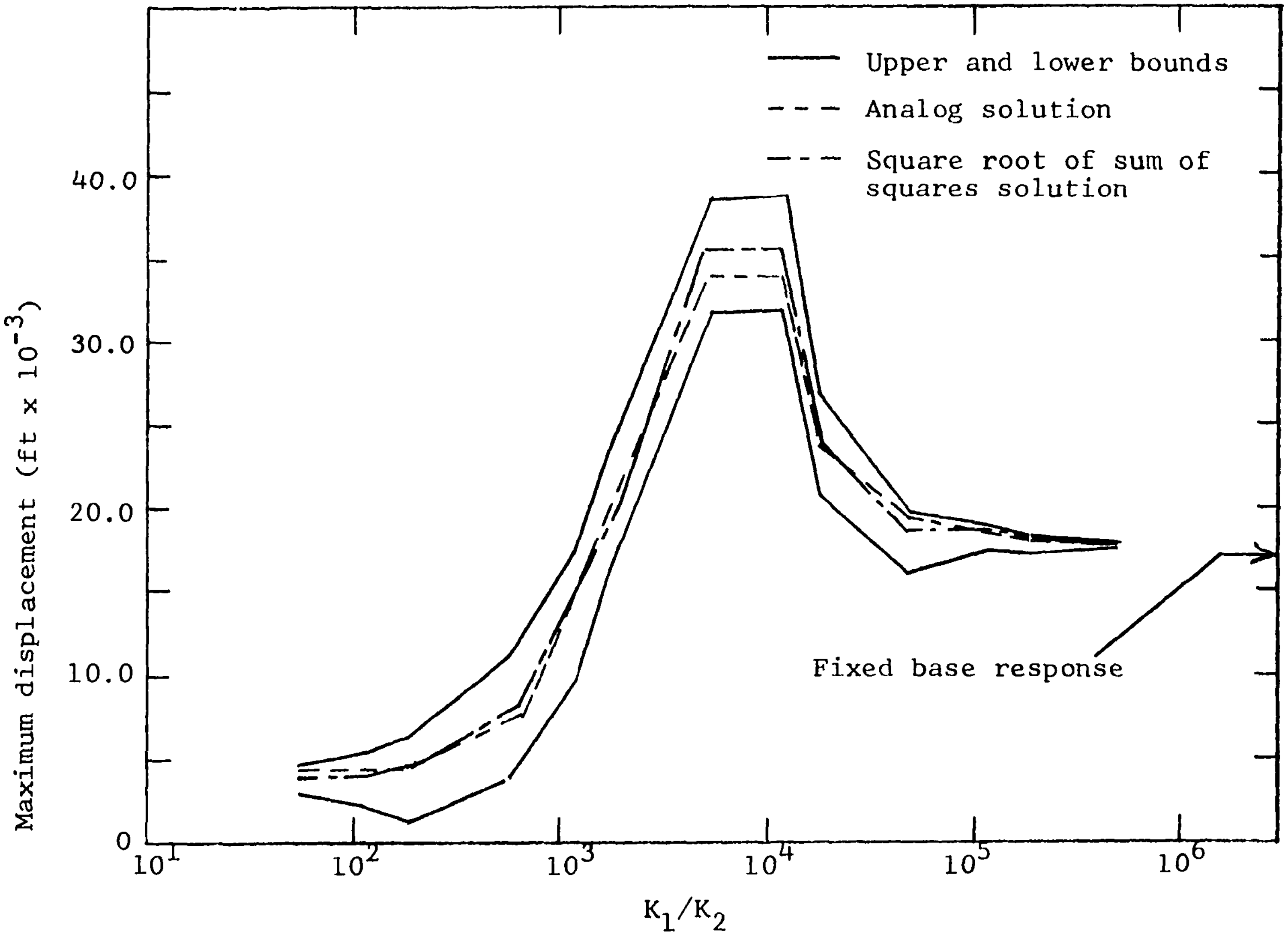


Figure 53. Type II rotational structure comparison of analog maximum displacement with that of various modal computed displacement maxima

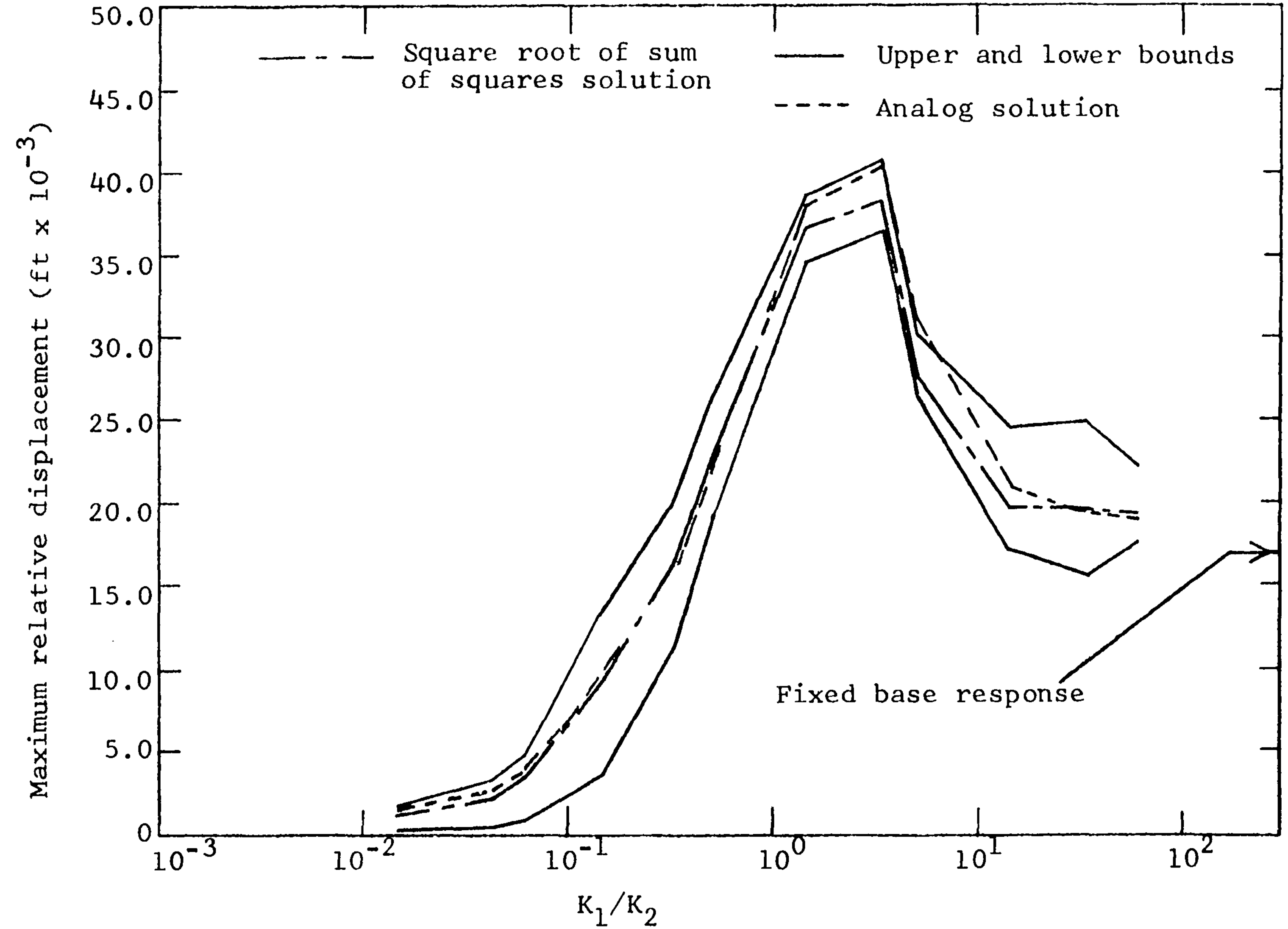


Figure 54. Type II combined variable K_r structure comparison of analog maximum displacement with that of various modal computed displacement maxima

damping on the structural displacement of a type I structure with a translational foundation is shown in figure fifty-five. Likewise, the effect of foundation damping on a type II structure with the same type of foundation is graphed in figure fifty-six. The influence of foundation damping on the structural displacement of type I and type II structures with rotational foundations is graphed in figures fifty-seven and fifty-eight and similar information on these structures where their foundations are combined foundations is shown in figures fifty-nine and sixty.

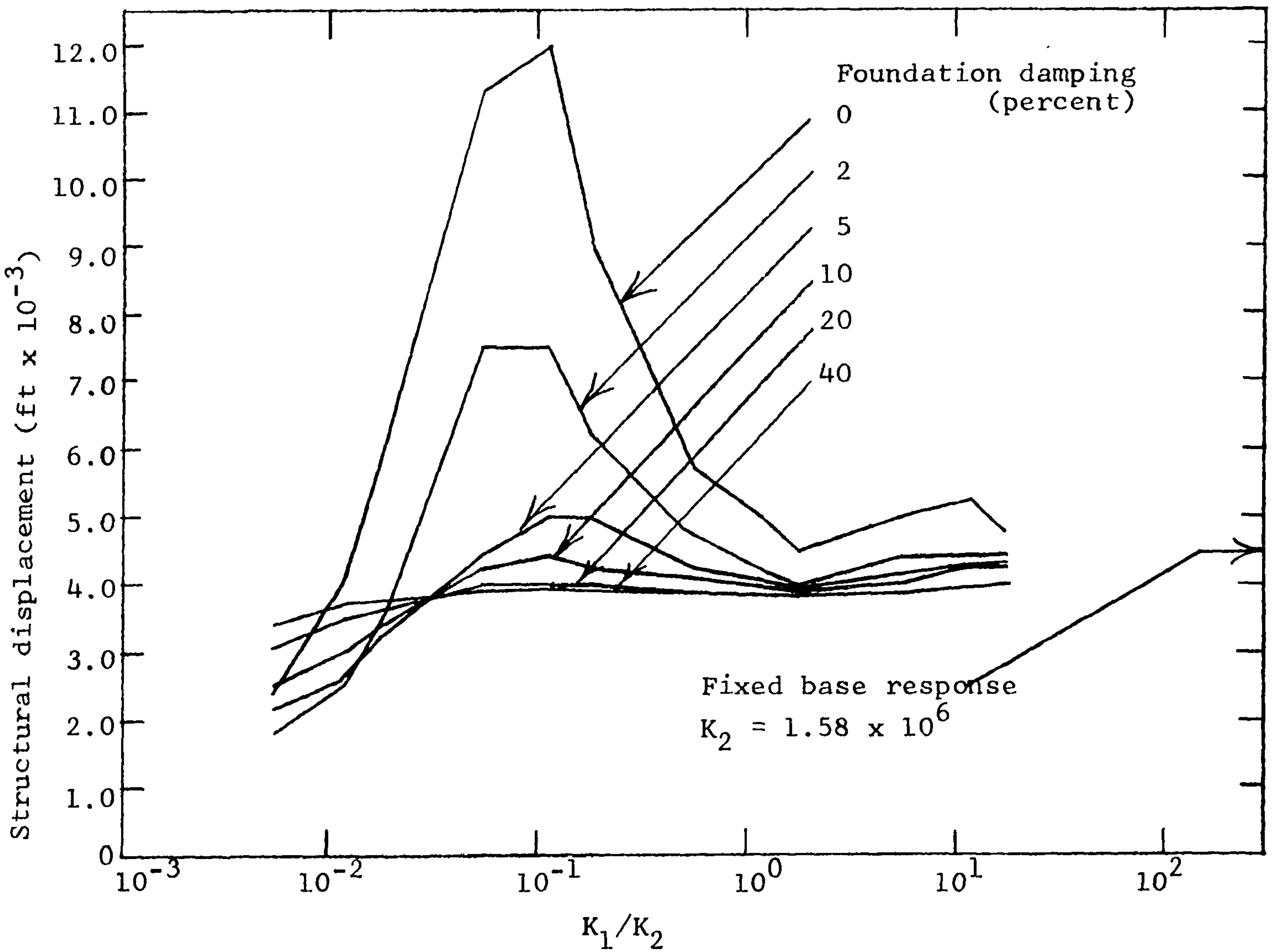


Figure 55. Displacement as a function of stiffness ratio for varied percentages of foundation damping for a type I translational structure

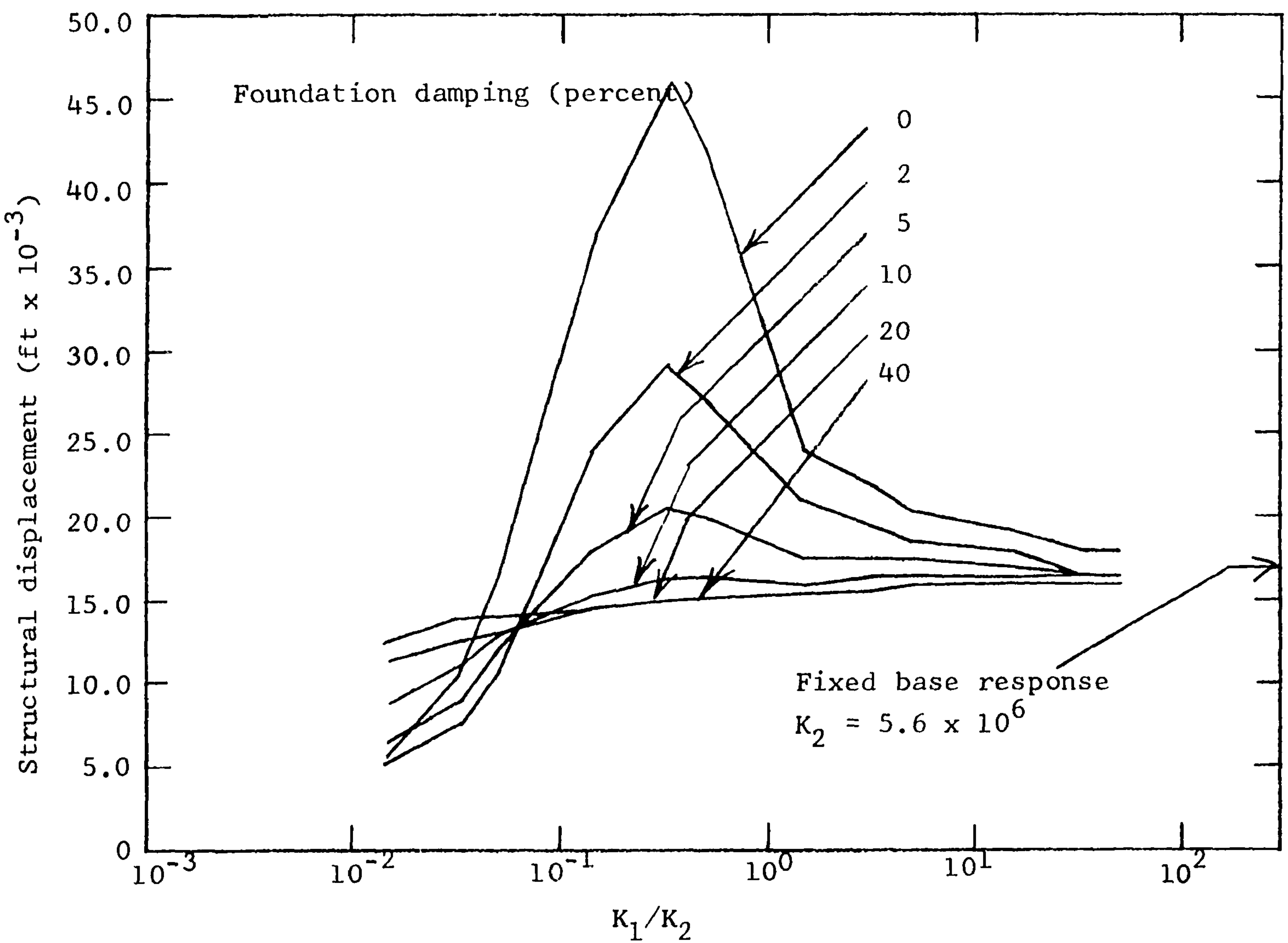


Figure 56. Displacement as a function of stiffness ratio for varied percentages of foundation damping for a type II translational structure

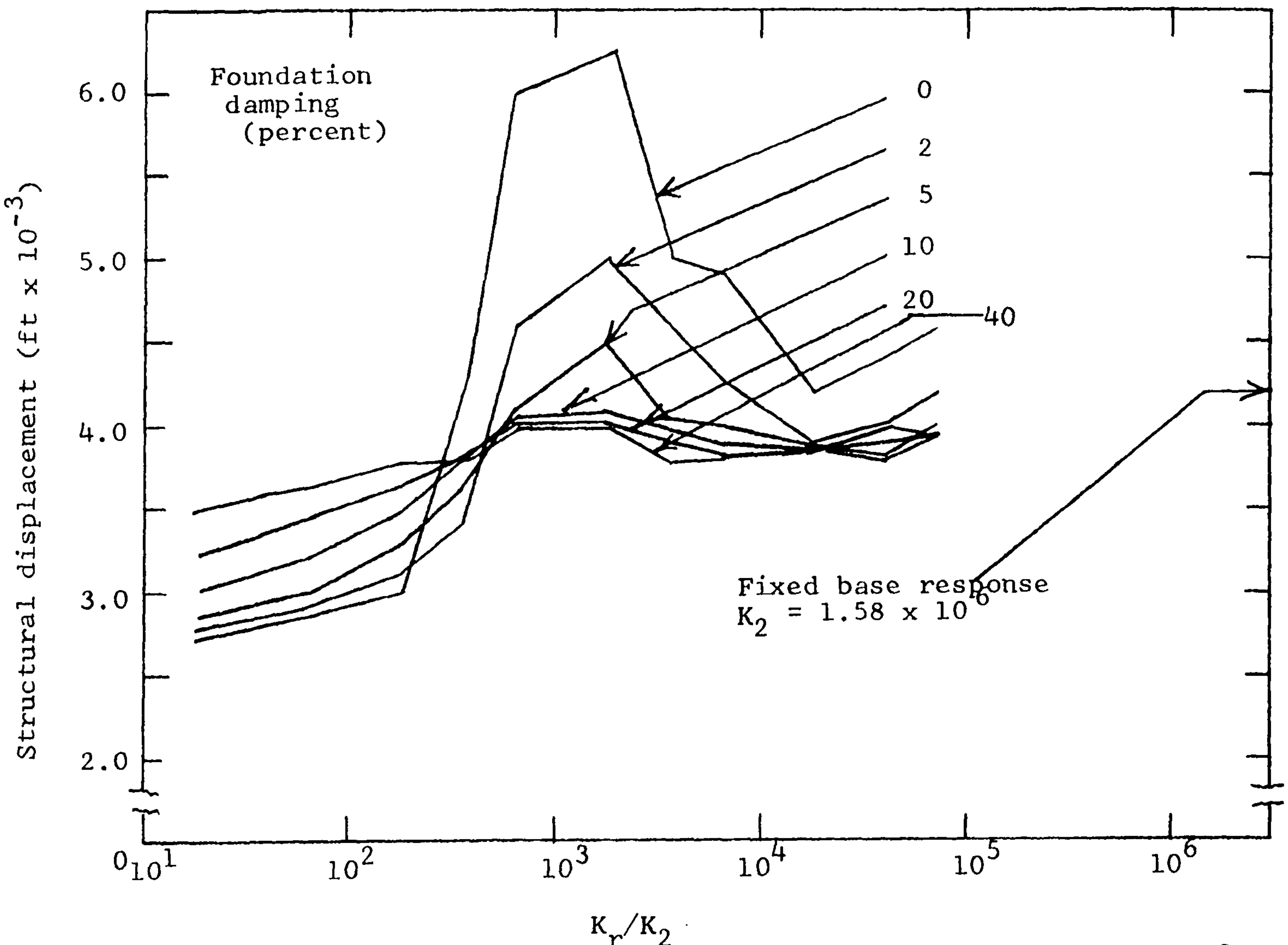


Figure 57. Displacement as a function of stiffness ratio for varied percentages of foundation damping for type I rotational structure

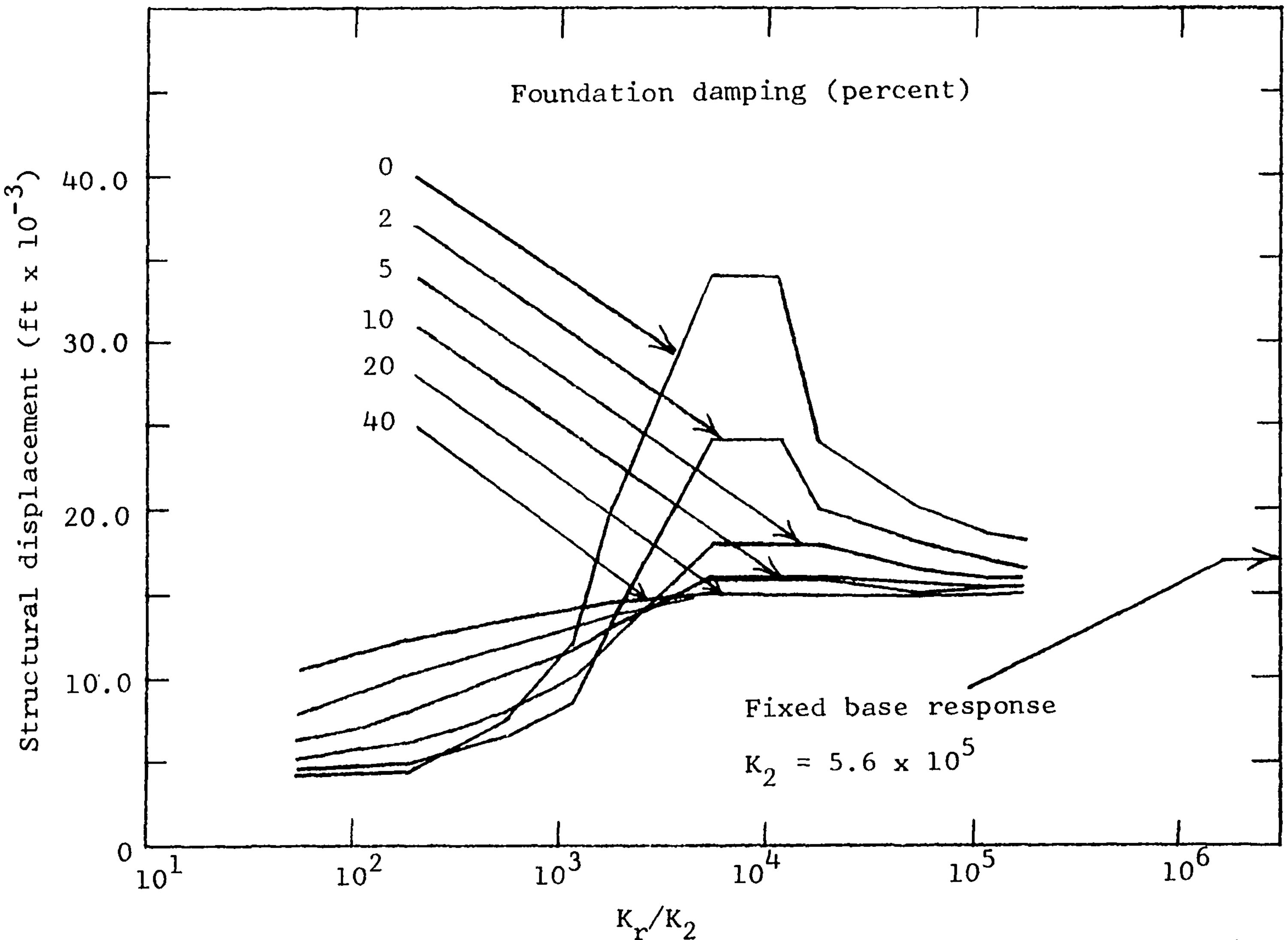


Figure 58. Displacement as a function of stiffness ratio for varied percentages of foundation damping for a type II rotational structure

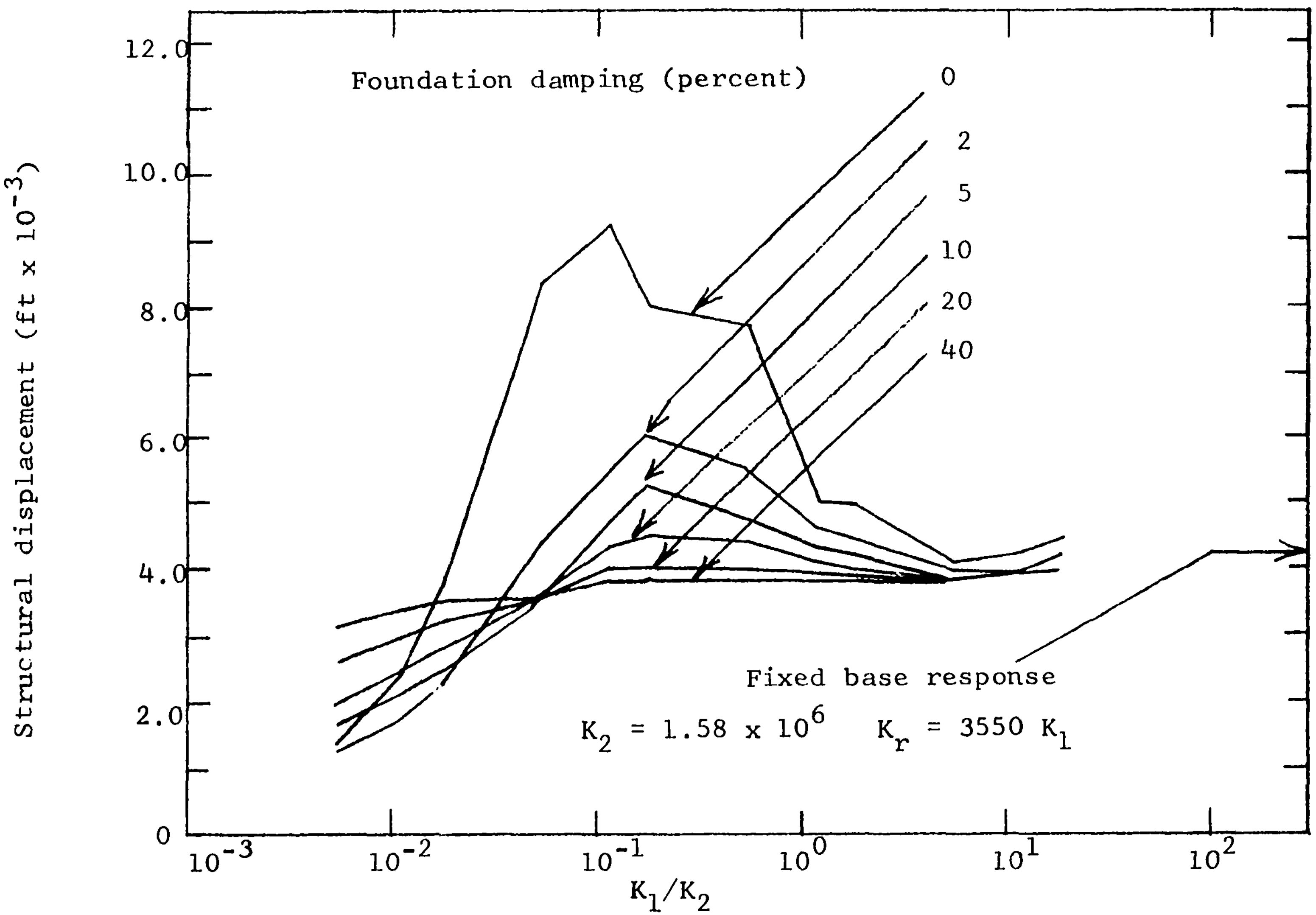


Figure 59. Displacement as a function of stiffness ratio for varied percentages of foundation damping for a type I combined variable K_r structure

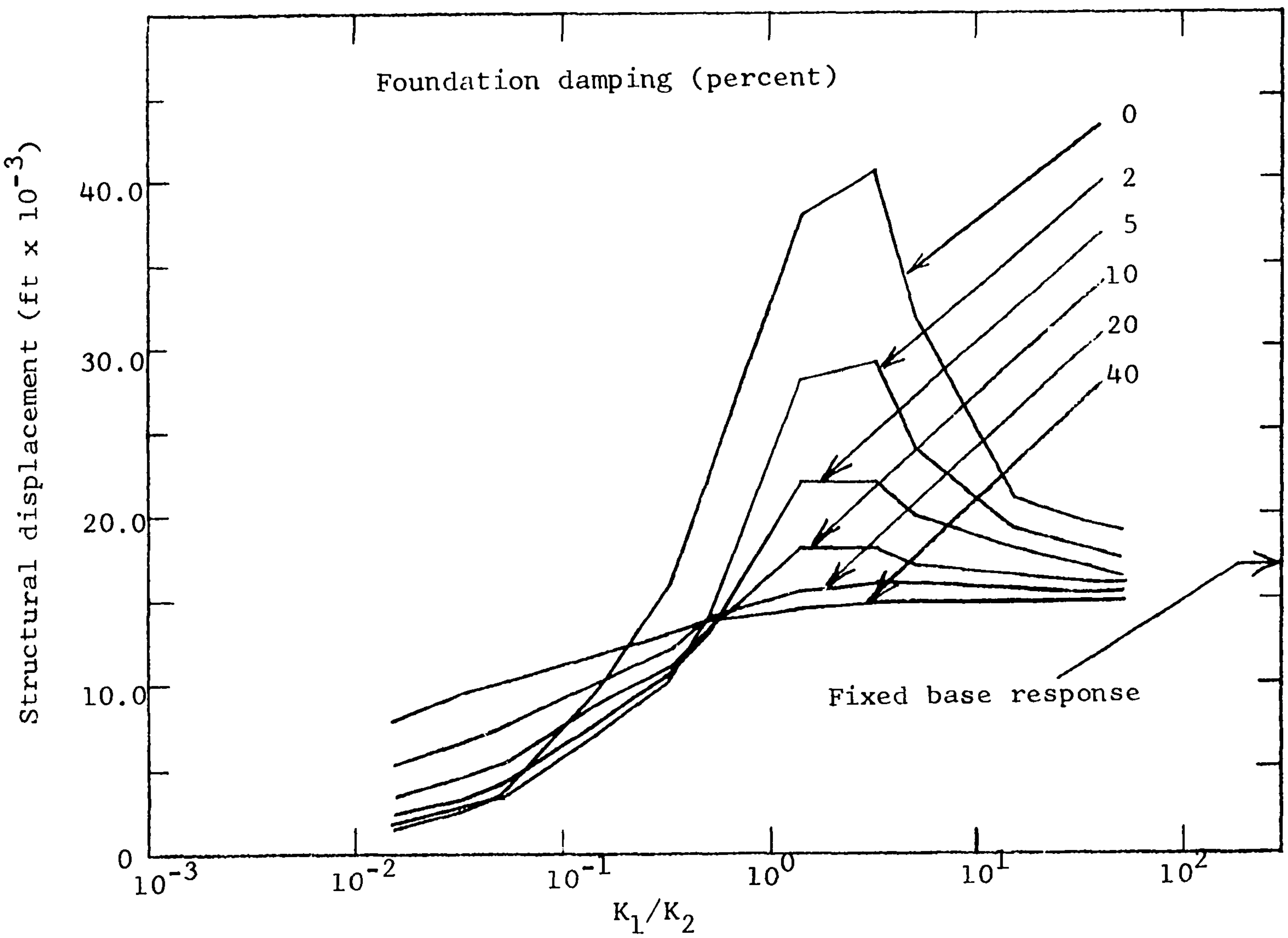


Figure 60. Displacement as a function of stiffness ratio for various percentages of foundation damping for a type II combined variable K_r structure

X. DISCUSSION

A. Foundation Stiffness Influence on Modal Frequencies

The usefulness of information regarding the undamped free vibrational modal frequencies can be observed by examination of response spectra such as the spectra shown in figures twelve and thirteen of this investigation. Referring specifically to figure thirteen as an example of a typical earthquake response spectrum it is observable that there exists a peak response in this El Centro undamped spectrum at around 0.35 cycles per second and a pronounced decrease in response beginning at about 1.5 cycles per second. Such response behavior is typical of all earthquake response spectra although the locations of response peaks and marked falloff of response varies. Knowledge of structural modal frequencies, then, provides an indication of how a particular structure will respond under seismic loading. Structures with modal frequencies in the vicinity of response peaks can be expected to show large response while structures with modal frequencies away from the zone of large spectrum response will be relatively unaffected.

Considering specifically containment structures and their modal characteristics, the fundamental mode frequency of a fixed base containment structure is, in general, found between 2.0 and 10.0 cycles per second. The structures identified as type I and type II structures for the purpose of this investigation have been calculated to have fixed base frequencies of about 8.3 and 4.1 cycles per second, respectively.

The general effect of altering the structural foundation from a bedrock or fixed base condition to that of a softer foundation can be seen from inspection of the curves in figures nineteen through twenty-two. A foundation approaching bedrock corresponds in these figures to K_1/K_2 and K_r/K_2 ratios of around 150 and 10^6, respectively. Transition to a softer foundation results in a shift to the left on these curves. As can be noted from these curves any reduction of K_1/K_2, for example, reduces higher mode frequencies. Reduction of K_1/K_2 to a value below about 5.0 results in rapid reduction of first mode frequency, also.

It can be seen from inspection of figure thirteen that modal frequencies of 2.0 and greater lie to the right of the response spectrum peaks. The reduction of frequencies as the structure is supported on a softer type of foundation, then, results in moving of the structure into a zone of increased spectrum response. One may, therefore, qualitatively expect that the structural response would increase as modal frequencies are reduced. This is not entirely the case, however. It must be pointed out that, as was developed in section VIII, the maximum modal response is a product of the response spectrum value times a fixed participation factor coefficient times the relative modal structural displacement. The mode shapes and hence the relative modal structural displacements vary as the relative magnitudes of foundation and structural stiffnesses vary. The mode shapes can, therefore, physically be thought of as measure of transmissibility of force into the structure. The mode shapes show relatively greater modal displacement in the foundation as foundation stiffness decreases. This effect is one of contributing to decrease in structural response as foundation stiffness decreases. The nature of the result is, consequently, a combination of these conflicting influences.

Turning now to a specific discussion of the information on frequency change found in figures nineteen and twenty, it is observable that at large K_1/K_2 ratios ($K_1/K_2 > 100.0$) the fundamental frequency of the structure does not change. This can be interpreted as indicating that the structure is, for K_1/K_2 values greater than 100.0, dynamically fixed based for the two degree-of-freedom situation considered in these figures. The higher frequency is in the range of 25 to 40 cycles per second which is out of the range of practical significance for earthquake spectra.

As the stiffness of the foundation (rocking or translational) is reduced the first mode frequency gradually reduces while the second mode frequency rapidly reduces at first. At about a K_1/K_2 ratio of 2.0, however, the second mode frequency levels out to a constant value and is unaffected by any further reduction in foundation stiffness. The reduction in foundation translational stiffness is slightly more effective in reducing the structural first mode frequency while rocking stiffness reduction appears slightly more effective in reducing the second mode frequency.

Reviewing the three degree-of-freedom systems with both foundation rotational and translational freedom the resulting modal frequency changes are depicted in figures twenty-one and twenty-two. The first and third modes of these systems (combined constant K_r and combined variable K_r) perform in a similar manner to the first and second modes of the previously discussed two mode systems. In fact, the first and second modal frequencies of the translational structure lie very close to the first and third modal frequencies of the combined variable K_r structure throughout the entire range of K_1/K_2 ratios.

The second mode of the three degree-of-freedom system behaves somewhat differently, however. For the combined variable K_r case, at large K_1/K_2 ratios, the second mode frequency takes on a very large value showing the essentially fixed base quality of this system at large K_1/K_2 ratios. As the K_1/K_2 ratio becomes smaller, however, this second mode frequency reduces rapidly and becomes closely parallel to the first mode.

Going onto the combined constant K_r case, the second modal frequency can be seen to become nearly constant at large K_1/K_2 ratios showing the essentially two mode character of this system at large K_1/K_2 ratios. As the K_1/K_2 ratio becomes smaller, however, this second mode frequency is seen to reduce rapidly at first and then become nearly constant again at K_1/K_2 values of about 0.5 and below.

To summarize the results of this section, it can be seen by the discussion that a rather significant change in the calculated value of modal frequencies can result when foundation properties are considered in what heretofore was a fixed base structure. It is practically impossible, therefore, to obtain a realistic estimate of structural frequencies of containment structures without knowledge of the foundation on which it is being constructed.

B. Foundation Stiffness Influence on Translational Structure Response

It is to be noted in referring to figures twenty-three through twenty-seven that the structural behavior of all translational containment structures when subjected to a bilinearly approximated El Centro spectrum are qualitatively the same. All structures exhibit a fixed base response at a K_1/K_2 ratio of 200 (not viewable in the figures). The response rises monotonically to a resonance response peak at a K_1/K_2 ratio of about 5.0 and subsequently shows a large decline. At a K_1/K_2 of about 0.1 the response of the structure is down to about twenty-five percent of its fixed base response.

The responses of type I and type II translational structures to another ground motion are shown in figures forty-nine and fifty-two. The response spectrum used to obtain these values is shown in figure twelve. The structural response to this alternate response spectrum is qualitatively quite similar to that response previously exhibited when the El Centro spectrum was used. Both of these structures also exhibit a fixed base response at a K_1/K_2 ratio of 200, rise to resonance peak structural responses at a K_1/K_2 ratio of about 5.0, and show a considerable subfixed base response at a K_1/K_2 ratio of about 0.01. A noticeable difference in response is the additional resonance peak at a K_1/K_2 ratio of between 0.05 and 0.5.

A comparison of structural response values achieved by each of these structures as each structure goes through its resonance is of interest. For all structures and both spectra all structures show a maximum structural response of between 175 and 200 percent of the fixed base response. This response is reached at a K_1/K_2 ratio of about 5.0. However, the second ground motion spectrum shows an additional and higher sum of modal maxima of about 2.75 times the fixed base response at a K_1/K_2 ratio of between 0.1 and 0.5. The occurrence of this second peak is thought to be a feature of the simple sine function used as ground acceleration and is not considered to be representative of response spectra in general.

The foregoing analysis of the results of these translational structure studies emphasizes the importance of proper modeling of the foundation. The assumption of a fixed base for a containment structure where, in reality, an appreciable amount of translational movement is possible, perhaps by placement of cushioning material between a bedrock layer and the containment base slab, when a translational structure should have been assumed, could result in design of a structure for lateral forces of only one-half of their actual magnitude. The assumption of a fixed based structure is not necessarily a conservative one, indeed.

The results also emphasize the importance of a reasonably accurate determination of foundation translational stiffness properties. Order of magnitude or rule-of-thumb estimates of the foundation translational stiffness will not suffice if any accuracy is to be gained from the analysis. It can be seen from the figures discussed in this section that an order of magnitude error in stiffness can result in 100 percent error in response. On the other hand if the translational stiffness is known to an accuracy of $\pm$ 15 percent, perhaps not an unreasonable requirement even for a material as variable as soil, a response of within 25 percent of the true response should be possible. Considering the uncertainty in knowledge of the ground motion, itself, this is about all one can require in accuracy at this point.

C. Foundational Stiffness Influence on Rotational Structural Response

The response of rotationally based containment structures to a bilinearly approximated El Centro response spectrum is presented in figures twenty-eight through thirty-two. All structures exhibit a fixed base response when the K_r/K_2 ratio is greater than 10^6 (not shown on the figures). As the foundation becomes softer and the stiffness ratio decreases the response falls of monotonically at a rather slow rate.

The resonance peaking assocated previously with the translational displacement does not occur. Since this study was based on the use of only one rather approximate response spectrum, however, it should not be inferred that this result is necessarily a general one. That can only be determined through analysis using a large number of response spectra.

The falloff in structural response with decrease in the rotational stiffness ratio (K_r/K_2) is interesting. It does not appear that reduction in the rotational stiffness ratio, i.e. reduction in foundation rotational stiffness, is as effective in reducing structural response as is a corresponding reduction in translational stiffness. In most cases the structural response remains between twenty-five and seventy-five percent of the fixed base response for very soft rocking foundations. For the corresponding very soft K_1 value of a translational (or combined) structure the response is always in the range of ten to twenty percent.

By way of summary, then, it has been seen in the previous discussion that the effect of rocking, as is developed in a rotational foundation as the rotational stiffness is decreased, is one of decrease in the response of the structure when compared with the response of a fixed base structure of the same dimensions and for the same ground motion input. The response reduction is very gradual, however, and considerable error is allowable in the assignment of a rocking stiffness value without appreciably effecting the structural response results.

D. Foundation Stiffness Influence on Combined Structure Response

The response of containment structures with both rotational and translational freedom to a bilinearly approximated El Centro spectrum is presented in figures thirty-five through thirty-nine and figures forty through forty-four. Figures thirty-five through thirty-nine consider the combined variable K_r case and figures forty through forty-four consider the combined constant K_r case.

Turning to figures thirty-five through thirty-nine it is observable that the general curve shapes are similar to the structural displacement and structural response curves graphed from translational structure calculations. Again, at high K_1/K_2 ratios the response is fixed based, as the stiffness ratio (K_1/K_2) is reduced a resonance effect occurs and, continuing on to even smaller values of K_1/K_2, the response and displacement are depressed to considerably below the fixed base values. Some explainable refinements are present, however. The peak response has been reduced slightly and shifted to a somewhat higher K_1/K_2 ratio. Where the maximum structural response occurred, previously at a K_1/K_2 ratio of 5.0 for the translational structure case, it now occurs at a K_1/K_2 ratio of about 10.0. There is a small response reduction to the low K_1/K_2 side of the peak response value.

This difference between the combined variable K_r structural response and translational structure response can easily be explained as the added effect of rocking action. Recalling from the rotational structure structural response graphs that the rocking response was below the fixed base value throughout its entire K_r/K_2 range. It was somewhat further below its fixed base value at low K_r/K_2 values than at large K_r/K_2 values. The combined variable K_r response, then, is essentially a synthesis of features of both translational and rocking behavior as one might expect.

Continuing on to the structural displacement and structural response values shown in figures forty through forty-four, the general curve features are very similar to the features just detailed for the combined variable K_r structure curves. Two differences become apparent from study of the curves, however. First, the structural response does not approach the fixed base response of the structure at large values of K_1/K_2 because rotational stiffness remains constant. Instead, at large values of K_1/K_2 the structural response approaches the structural response value corresponding to the specified K_r/K_1 value of the rotational structure case.

Second, to the left of the resonance peak the structural displacement and response do not fall off monotonically. Rather, a second small resonance peak is in evidence at a K_1/K_2 ratio of about 0.5. This resonance peak is attributable to increased second mode contribution to response in the region. It, also, has the effect of increasing the structural response somewhat in the low K_1/K_2 ratio region below this additional resonance.

E. Effect of Mass Partitioning on Structural Response

An extremely accurate approximation of mass distribution was not required for the purpose of this study inasmuch as the effect of foundation on overall structural response and not the details of the structural action, itself, was the principal interest. Also, the limited number of analog components (operational amplifiers) available dictated that the structural mass distribution be kept as simple as possible. Hence, a two mass approximation of the containment mass was adopted. It was the simpliest mass distribution that would still describe the essential features of the problem.

As indicated previously (section VI) four different mass partitions were used and the effect of each of these partitions on the problem was determined for type I and type II structures. For the remaining type III, IV and V structures only one mass distribution, the I^1 equated mass distribution, was used.

The effect of each of these mass distribution assumptions can be observed by referring to figures twenty-three through twenty-six, twenty-eight through thirty-one, thirty-five through thirty-eight, and forty through forty-three. Referring to figures twenty-three and twenty-four the effect of making these various mass partition assumptions can be observed for a type I structure with translational foundation. The effect of increasing the portion of the sidewall mass in the top lumped mass is seen as one of raising the absolute displacement slightly. The structural response is decreased slightly by this mass change. Overall, changing the top mass concentration from that of including only one-third of the sidewall mass to that of including two-thirds of the sidewall mass results in a change of response of ten percent of the fixed base response for low K_1/K_2 values to thirty percent for a K_1/K_2 value of about 5.0. The effect of this mass variation on a type II structure with a translational foundation is very nearly the same so it will not be discussed in detail.

Turning to the effect that variation in the mass partition can make in the response of a rotational type I structure the results are shown in figures twenty-eight and twenty-nine. The result in increasing the top mass from one of including one-third of the sidewall mass to one of including two-thirds of the sidewall mass is a rather constant ten percent increase in structural response. The effect of mass partition variation on a type II rotational structure is much the same. A constant approximate twenty percent change is evident, however.

The influence of mass variation on the structural response of type I and type II combined variable K_r structures are shown in figures thirty-five through thirty-eight. For both type structures it is observable that the effect of increasing the top mass is one of both increasing the maximum response and shifting it to a lower K_1/K_2 ratio. The effect on structural response is just the reverse, i.e. the structural response is decreased and shifted to a higher K_1/K_2 ratio. The effect of mass variation for the combined constant K_r structure is essentially the same as for the combined variable K_r situation. Hence, it will not be discussed in detail.

The comparative effect that various mass partition assumptions have on the structural response for structures of differing height can be seen by comparing against each other the type I and type II structure response changes cause by the variation in mass partition for a given foundation condition. It is observable that very little effect of height is noticeable for the translational and combined constant K_r foundations. However, when the rocking and combined variable K_r structures are considered, the taller type II structure shows a somewhat increased sensitivity in structural response by the mass partitioning used.

By way of summary, it is obvious that even fairly extreme assumptions as to effective mass partition do not change very greatly the structural displacement and structural response values obtained from the analyses. The assumption of mass partition does not affect in any way the conclusions of the stiffness part of this investigation since they are mainly qualitative.

The insensitivity of structural response to mass assumption may be regarded in a broad sense as indicating that the continuous structure response is fairly accurately represented quantitatively by the two lump mass approximation. For the lumped mass/continuous mass structure with a fixed base accurate modal frequency information is available. The first mode frequency of 7.94 compares favorably with the 8.18 fixed base frequency obtained from the half sidewall lumped top mass. The one-third and two-thirds sidewall top mass approximations have frequencies of 8.86 and 7.62, respectively. Inasmuch as half sidewall lumped mass approximation represents well the true first mode frequency for the fixed base case and the one-third and two-thirds mass approximations from rather wide frequency bounds, it is a logical deduction that these three mass approximations represent well the best approximate and bounding values in the actual structural response of containment structures throughout the entire foundation stiffness range.

F. Omission of Component Mass and Moment of Inertia Effect on Structural Response

The effect of omission of component mass and/or moment of inertia on a type II structure is viewable in figures forty-five through forty-eight for various foundation conditions. Several effect, some of them fairly obvious, are noteworthy. For a translational structure it is obvious that internal moment of inertia should have no effect on the resulting structural response and the calculated results confirm this observation. Likewise, for a rotational structure omission of the internal mass contribution to the base mass should not have any influence on structural response. That this is the case can be seen in figure forty-six.

Change in the base mass magnitude of a type II translational structure through complete neglect of the mass of internal structures and equipment does change, but not significantly. its structural response except in the region of resonance ($2.0 < k_1/K_2 < 50.0$). In this resonance region the response is reduced from 175 percent to 150 percent by this change. For the rotational type II structure the complete neglect of the internal strucutres and equipment contribution to the total mass moment of inertia results in a small but fairly constant ten to fifteen percent change in structural response throughout the entire range of stiffness ratios.

Turning to the combined type II structure cases as shown in figures forty-seven and forty-eight total change in structural response considering change in mass and/or moment of inertia with change in stiffness ratio is somewhat more complex. Total neglect of mass and/or moment of inertia of internals does not affect the structural response greatly for K_1/K_2 ratios less than 1.0 or greater than 40. However, in the resonance range omission of either mass and/or moment of inertia can be seen to reduce the resonance response peak from an estimated 200 percent of fixed base response to an estimated 175 percent of fixed base response.

To summarize the above discussion, it is seen that complete neglect of mass and/or moment of inertia of structural internals does not have a great effect on structural response computations. For the designer, an estimate on the moment of inertia of his structures and internals to no less accuracy than $\pm$ 25 percent should be available from previous projects. With information on internal structure and equipment mass and moment of inertia available to this accuracy, the computation of overall containment structural response should be well below the accuracy currently permitted by uncertainity in seismic ground motion.

G. Effect of Structure Height on Ground Motion Developed Structural Stress

Consider the influence that a containment structure's height will have on its response under ground motion. The difference between the response of a short type I translational structure and that of a taller type II translational structure can best be seen by comparing the structural displacements developed in each structure at identical K_1 values. This comparison can be accomplished by comparing the values in figures twenty-three and twenty-four. It is to be observed from these figures that the taller, more flexible type II structure develops around twice the displacement of the shorter type I translational structure. Note that the abscissa is expressed in terms of K_1/K_2 values and, also, recall that the value of K_2 of the type I structure is about three times that of the taller type II structure. The displacement values, therefore, for type I structures should be compared with corresponding values of the type II structure at a K_1/K_2 abscissa of about three times larger.

The effect of height on the displacement of a rotationally based structure is observable from a comparison of the type I and type II structure displacement curves of figures twenty-eight and thirty. In magnitude of displacement and for a given K_r value, the taller structure again shows about twice the displacement of the shorter structure. The same precaution with regard to comparing displacements at equal K_r values and not equal K_r/K_2 ratios should, of course, be observed if an accurate point by point comparison is desired.

The above mentioned displacements are not a comparative measure of structural stress and strain when structures with different heights are considered. It is desirable that the stress levels in these structures be compared. The previously referenced graphs of structural displacement do not show this information, but it is easily shown that such a stress level comparison is actually being made if values of $K_2X_2H_2$ are compared. For the type I and type II structures discussed in this section, multiplying type II structure displacement values by a factor of 0.71 permits such a stress level comparison to be made. Figure sixty-one is a graph of such a stress level comparison of type I and type II translational structures. Figures sixty-two through sixty-four give the same stress comparisons for type I and type II structures for rotational, combined constant K_r and combined variable K_r foundation situations, respectively.

An inspection of these figures shows that the taller type II structure, with any of the types of bases considered, developes considerably greater structural stress than does a type I structure with a comparable base and identical section properties. This result is to be expected inasmuch as the type II structure has a much greater M_2 mass and mass moment. The type I curves are identical to the curves discussed previously while the type II curves are also identical to the previously discussed curves in shape but are reduced in magnitude by the previously cited factor of 0.71. Each of these curves shows the peak stress in the taller type II structure to be about 150 percent of the peak stress in a comparable type I structure. For specific values of K_1 the stress ratio varies widely, being as high as 400 percent for some values to as low as 110 percent for others.

It normally would be considered a logical move on the part of the structural designer to reduce his containment structure's height if seismic stress starts to significantly influence his design. Given a height reduction factor of two, an increase in twice the base area (and accompanying increase in K_r and/or K_1) would normally also be made due to a usual design requirement for a specific containment volume. In making such a change the designer would probably also strive to maintain pneumatic longitudinal and hoop stress unchanged. Under such conditions, the thickness and radius and, hence, the stiffness of the structure would be greatly increased.

The stress developed in a structure with an increased base area is, of course, proportional to its $K_2X_2H_2$ factor as was the case with the previous structures. The H_2 is, again, the structure's overall height. The K_2 value can be computed from a knowledge of the structure's height and section properties in the manner previously described. To obtain the X_2 value use was made of the fact that structural response for a given K_1/K_2 (or K_r/K_2) ratio is insensitive to K_2. Individual X_2 values were then obtained by multiplying the selected structural response ratio by the response spectrum value corresponding to the fixed base of the reduced height, increased base, structure.

The structural relative stress of such a modified structure with a translational foundation is shown as the dashed curve in figure sixty-one. It is seen in referring to this figure that the stress in the modified structure is greatly reduced as compared to stress developed in its companion type II structure. The dahsed curve reaches a peak of value of 23.3 stress units at a K_1 of 100 x 10^6 (not shown in this figure). The unit stress then drops off gradually at higher values of K_1. At a value of 10^8 the stress in the structure is essentially the fixed base stress value of 13.3 units.

Using the same factor of two height reduction on a type II structure, but with a rotational foundation situation, the resulting stress change can be seen in figure sixty-two. It is observable that the modified structure developes through the entire stiffness range only about twenty-five percent of the stress that is developed in the taller structure for the same ground motion.

For the combined structure cases the general behavior will be qualitatively similar to the aforementioned translational

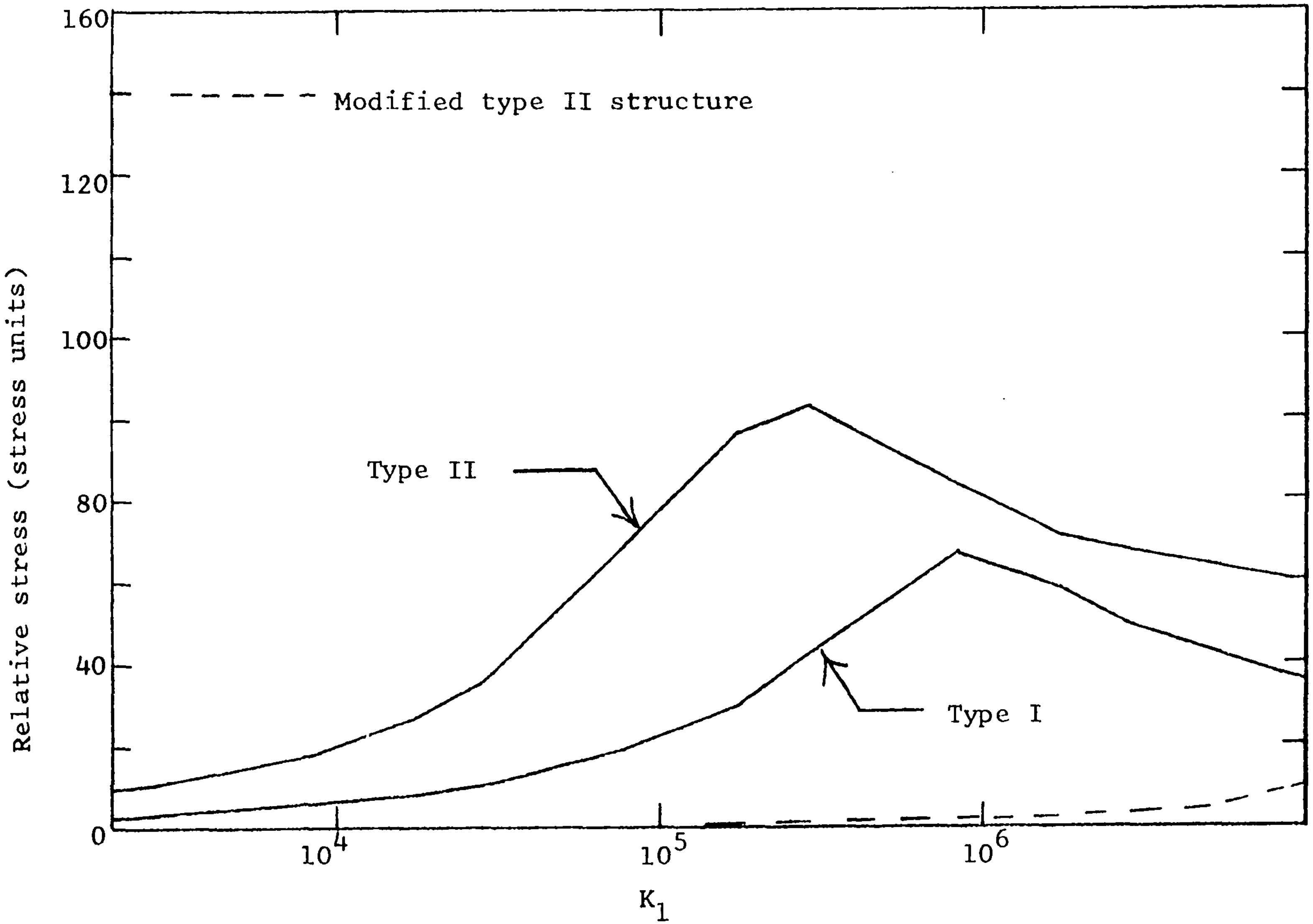


Figure 61. Comparison of stress in translational type I and type II structures as a function of foundation stiffness

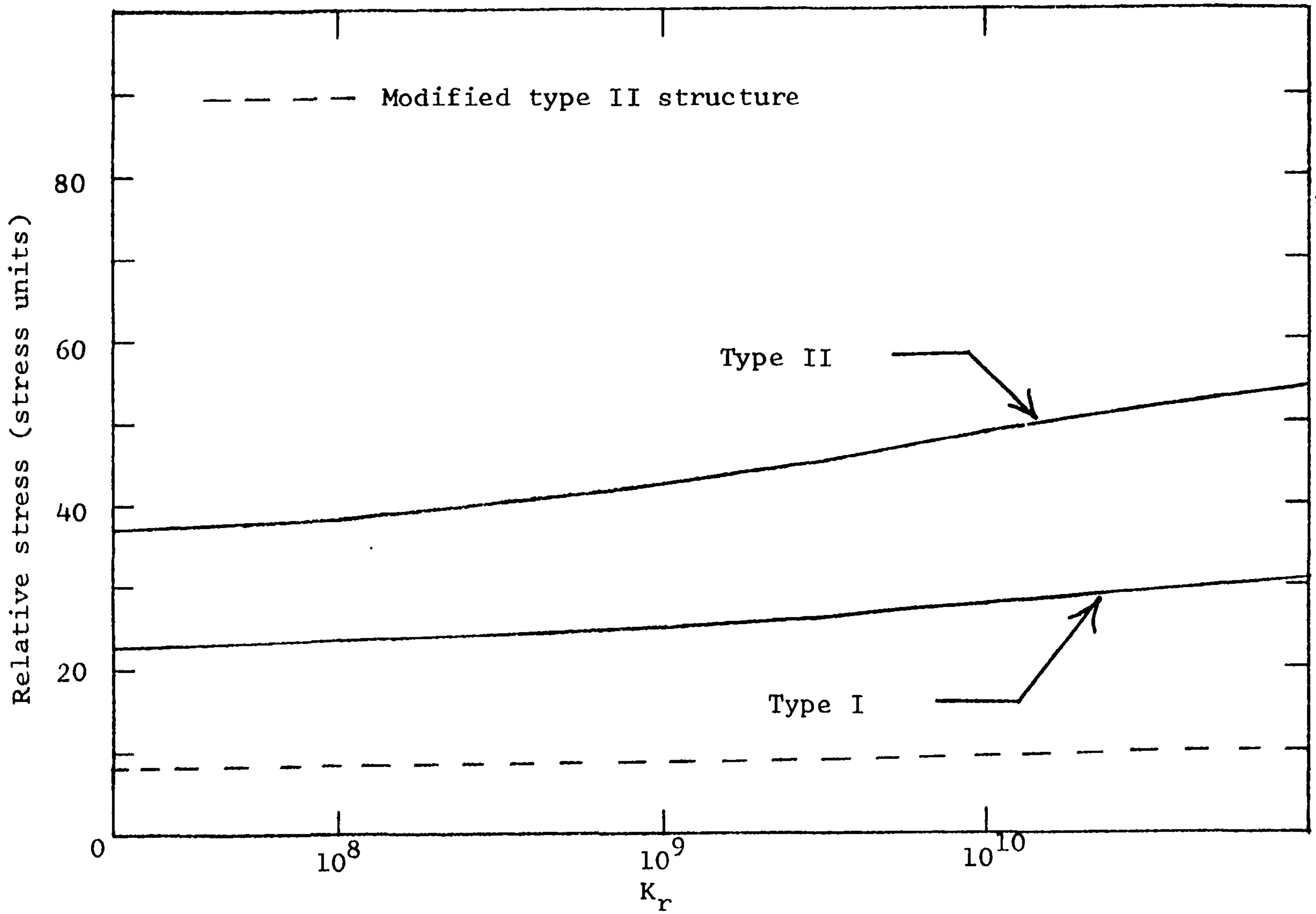


Figure 62. Comparison of stress in rotational type I and type II structures as a function of foundation stiffness

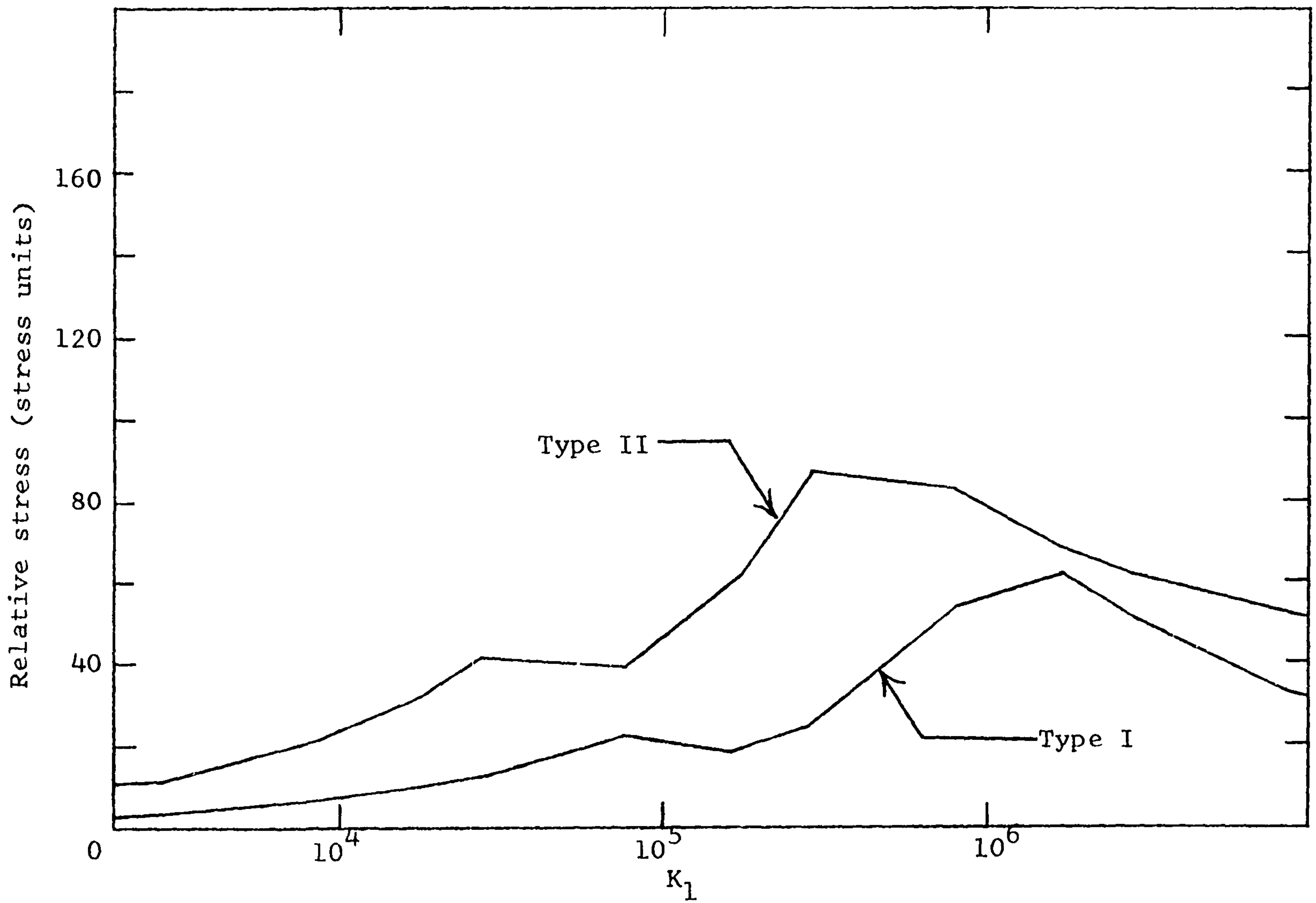


Figure 63. Comparison of stress in combined constant K_r type I and type II structures as a foundation stiffness

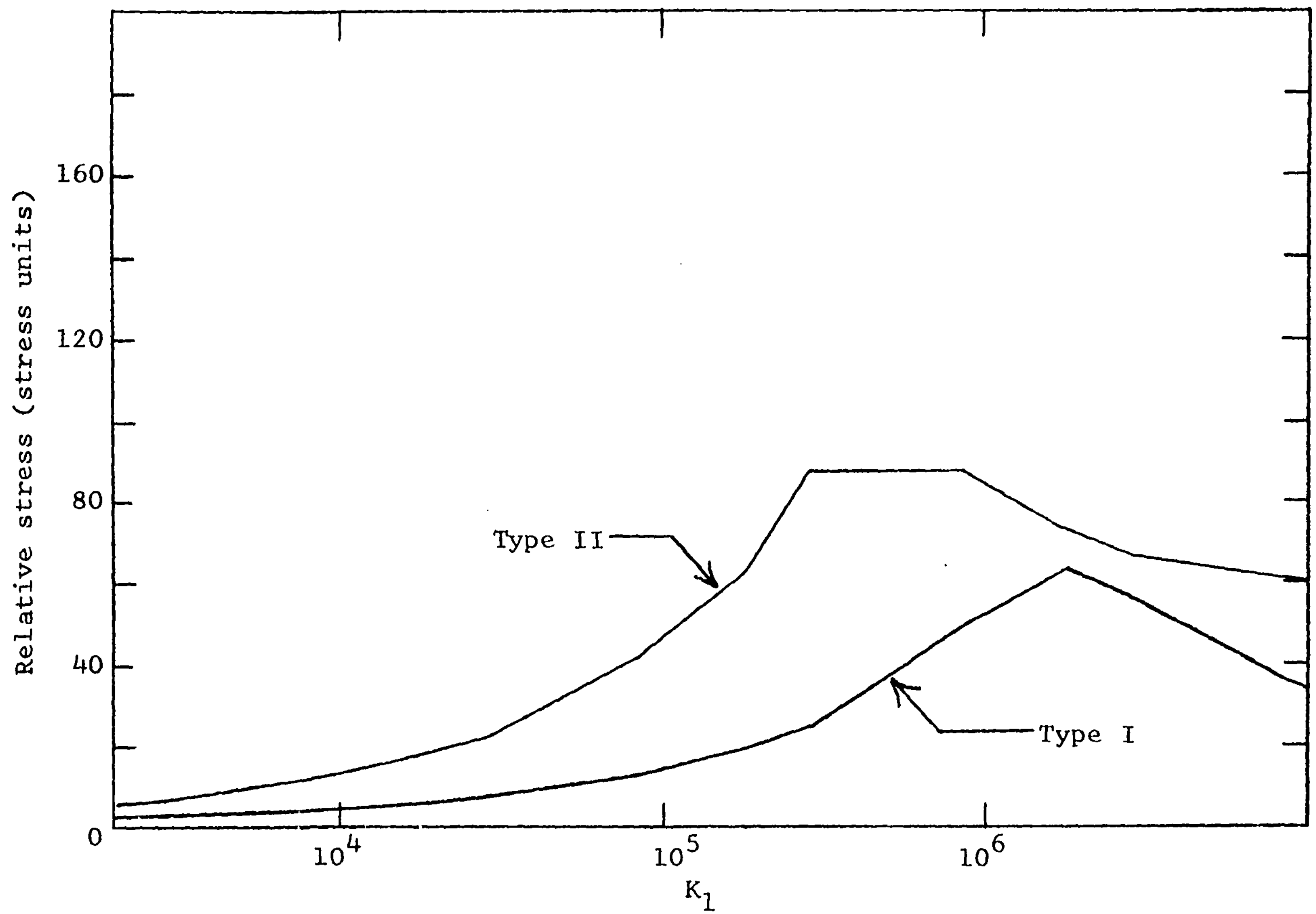


Figure 64. Comparison of stress in combined variable K_r type I and type II structures as a function of foundation stiffness

structure behavior. However, the stress reduction for these cases cannot be computed since different ratios of K_l/K_r are involved than those that are available from the results of this study.

H. Reliability of Analog Results

As an initial verification of the correctness and accuracy of the analog patching for the damping portion of this study, a series of modal analyses were made for the zero damping case using the response spectrum of figure twelve. An exact digital modal analysis solution is not obtainable. However, the absolute modal sum gives an upper bound on the correct solution, the square root of the sum of modal maxima gives a general average value of what the solution may be and, of course, the absolute difference between two modal maxima gives a lower bound for the correct solution. These modal combinations results, along with the analog developed solutions for the same series of problems, are plotted for the type I and type II translational, rotational and combined variable K_r structures in figures forty-nine through fifty-four.

For the analog computer set up to be a valid representation of the particular problem being considered the analog determined maximum must lie between the upper and lower bounds established by a modal analysis solution to the same problem. That this qualification has been met can be observed by scaning the above cited figures.

The general accuracy of the analog analyses can also be inferred from these figures by comparison of the closeness of fit with which the analog maximum corresponds to the upper and lower bounds at pinch points, i.e. points where the maximum and minimum bounding values are close together, the very good fit that has been achieved at such points is indicative of the accuracy of the analog solution. The analog recorder is readable to an accuracy of between one and three percent, depending on the strength of the output signal and scale used. The electronic components of the analog have order of magnitude greater accuracy. The accuracy of the results shown in these figures is, therefore, the accuracy to which the analog recorder is readable, namely one to three percent.

I. Comparison of Digital and Analog Results

Figures forty-nine through fifty-four described in the previous discussion relating to analog computer set up validity are essentially graphs of exact maximum structural displacement values (analog values) and maximum structural displacement values obtained by the use of various modal combination criteria. While not the main thrust of this study, it is, nevertheless, of interest to compare these results. It has been suggested by Clough (8), for example, that, for a small number of modes, the most appropriate modal combination to use to approximate the true maximum is the sum of absolute modal maxima whereas for a large number of modes he suggests that the use of a square root of the sum of the squares of modal maximum will yield a closer approximation to the true maximum.

For the six cases studied (see figures forty-nine through fifty-four) the actual (true) response was quite varied as can be observed from the figures. It varies from below the computed square root of the sum-of-the-squares value in figures forty-nine and fifty-two to that of following closely the sum of modal maxima for parts of the curve in figures fifty and fifty-one. For most of the cases considered, some sections of the true (analog computed) response versus stiffness ratio curve are at the sum of the modal maximum and other sections trace along the square root of the sum of the squares maximum. Along certain sections the true response also drops below the computed square root of the sum of the squares maximum, however.

These results seem quite scattered. To place them in perspective, however, it should be emphasized that these are results for only one forcing function. A slightly different time variation or duration of forcing function would undoubtedly cause the true value to take a different path so that in another section of the graph it approached an absolute maximum. Accepting such argument, it must be concluded that, for the two and three mode cases considered, the sum of absolute modal maxima is the logical best choice for the maximum value that will be attained. It is also the one that, of course, supports the previous suggestion by Clough.

J. Foundation Damping Influence on Structural Response

The effect of damping on the structural displacement of translational type I and type II structures is shown in figures fifty-five and fifty-six. It is observable in these figures that a) the undamped displacement of the type II structure is approximately four times that of the type I structure, b) the displacement of both type I and type II structures has reached a constant fixed base single degree-of-freedom value at a K_1/K_2 value of about fifty and greater, c) both structures show resonance peaking at intermediate values of K_1/K_2 ($0.05 < K_1/K_2 < 5.0$) with a resonance displacement in each case of about three times the fixed base displacement and d) the structural displacement drops considerably below the fixed base value at small K_1/K_2 ratios ($K_1/K_2 < 0.01$).

The effect of damping on structural displacement can be seen principally as that of decreasing the structural displacement from its undamped value in the region of resonance and as that of increasing the structural displacement over its undamped value at low K_1/K_2 ratios ($K_1/K_2 < 0.05$). A two percent foundation damping coefficient reduces the structural displacement increase above its fixed base value to one-half of the undamped structural displacement increase. Five percent foundation damping, considered at all unreasonable for a typical soil foundation, reduces this resonance increase in displacement to one-fourth of the displacement increase that occurs without foundation damping being present. Further increases in foundation damping continue this decrease in response change to such an extent that at forty percent damping the response is quite flat throughout the entire range of K_1/K_2 ratios of interest.

Turning to the rotationally based type I and type II structures, the general pattern of structural displacement is graphed in figures fifty-seven and fifty-eight. It is observable for the zero damping case that a) the maximum relative structural displacement of the type II structure is again about four times the displacement of the shorter type I structure, b) the response of both structures is essentially a constant, fixed base value at a K_r/K_2 ratio of about 10^5 and greater, c) both structures exhibit a resonance response that rises to a maximum displacement of about twice the fixed base displacement in the intermediate K_r/K_2 region and d) the displacement falls to considerably below the fixed base displacement at a K_r/K_2 ratio of 500 and below.

With respect to damping and again referring to figures fifty-seven and fifty-eight, it can be observed that damping substantially surpresses the resonance response for the rocking cases in the same manner that it affected the translational structure cases. A two percent foundation damping again reduces the increase in displacement over the fixed base value at resonance to one-half of the increase in displacement that would have occurred at resonance if no foundation damping had been present. A five percent foundation damping limits the resonance increase in displacement to one-fourth of the undamped displacement increase. Further increases in foundation damping percentage further suppress this resonance increase until at around forty percent damping the resonance is almost completely suppressed.

Considering now the effect of damping on combined variable K_r type I and type II structural displacement, the results are shown in figures fifty-nine and sixty. As might be anticipated the combined displacement represents somewhat the averaging of rotational and translational displacements. The relationship of the undamped displacement characteristics of the type I structure to the undamped displacement characteristics of the type II structure are the same as in the previous two cases. Likewise, the effect of damping on these displacements is essentially the same as that reported for the translational and rotational cases.

In summary, it has been shown that foundation damping can have a significant effect in limiting structural response. It bears emphasizing, however, that each of these damping cases were developed with two percent structural damping in the system in addition to the varying percentages of foundation damping. However, in all cases the analog problem was run initially with zero and two percent structural damping but with no foundation damping to obtain the zero foundation damping response. The values of these two cases were always essentially the same, i.e. there was virtually no effect on the peak structural displacement by use of two percent structural damping.

The curves for zero foundation damping are, therefore, really two superimposed curves; one for zero structural damping, and one for two percent structural damping.

K. Capability of Modal Analysis Technique to Describe Damped Structural Response

A specific objective of this investigation has been to evaluate the effectivness of the currently used modal analysis technique for predicting the structural response of containment structures with structural and foundation damping. With this view in mind the percentage response change for various percentages of modal damping was calculated for the previously selected series of foundation stiffness values. For each specific calculation the modal damping percentage was held to the same value for each mode. The response spectrum curve in figure twelve corresponding to this percentage of critical damping was used to determine the structural displacement factor. The results of these modal analyses have been plotted as an overlay to the actual analog evaluated response reduction percentages obtained when various selected amounts of foundation damping are used along with the same foundation stiffness values. The results of these comparative computations for type I and type II translational, rotational and combined variable K_r structures are shown in figures sixty-five through seventy. The modal ordinate values in these figures refer to the damped modal sum of maxima displacements as a percentage of the undamped sum of modal maxima. Likewise, the analog ordinate values in these figures refer to the damped analog computed displacement values as a percentage of the undamped analog displacement. It is to be recalled that the sum of modal maxima was shown in a previcus section to be a good representation of structural response for two and three degree-of-freedom systems such as considered in this study.

Discussing first the translational type I structure the results are graphed in figure sixty-five. At large and intermediate values of K_1/K_2 ($K_1/K_2 > 0.05$) it is observable that a value of five percent foundation damping corresponds reasonably well to forty percent modal damping. Likewise, a value of two percent foundation damping corresponds to between ten and twenty percent modal damping for this same K_1/K_2 range. Greater percentages of foundation damping than five percent do not appear to offer much by way of additional response reduction, however. An explanation for this observation that larger than five percent foundation damping does not offer much in the way of increased structural response reduction lines in the fact that the structure has been reduced to close to a fixed based structure already at five percent damping.

For values of K_1/K_2 in the region of less than 0.05 a noticeable deviation between modal evaluated and actual analog evaluated structural response occurs. Modal analysis predicts structural response to remain somewhat suppressed due to damping. However, analog analysis shows that at low K_1/K_2 ratios the response actually increases over the response the structure has without damping. This is interpreted as showing that the actual effect of increased damping in the low K_1/K_2 region is one of transferring more force into the structure and, hence, increasing structural response. Considering the value of two percent of foundation damping, it is seen that an equivalent ten percent foundation damping does a fairly effective job of modeling the damping effect down to a K_1/K_2 ratio of about 0.01. However, for a foundation that can be estimated to develop larger than two percent damping application of modal analysis even with the assumption of large damping percentages appears to give very poor results at K_1/K_2 ratios below 0.05.

Going on to the case of a type I structure with a rotational foundation the results are graphed in figure sixty-six. Again, a five percent damping of foundation rocking motion is the close equivalent of forty percent modal damping for intermediate and large K_r/K_2 ratios ($K_r/K_2 > 2000$). For the same range of K_r/K_2 ratios, a two percent foundation damping is seen as approximately equivalent to twenty percent modal damping. Also, as in the previous type I translational structure case, the modal analysis procedure is seen as not being capable of giving good structural response reduction predictions at low K_r/K_2 ratios ($K_r/K_2 < 2000$). The effect of essentially fixed base action occurring at foundation damping percentages greater than twenty percent is again observable.

Continuing on to the type I combined variable K_r structure the results are shown in figure sixty-seven. A comparison of this figure with figure sixty-five showing the results for the type I translational structure reveals a great similarity in response reduction. The previous discussion with reference to the type I translational structure is equally valid for the type I combined variable K_r case.

The results of the comparative analyses for the taller type II translational structure are shown in figure sixty-eight. The general appearance of both the modal and analog evaluated effects of damping on structural response are very similar to the previous results obtained for the type I translational structure. The one real difference appears in the range of applicability of modal analysis. The validity of modal analysis extends for this structure down to an approximate K_1/K_2 ratio of 0.03 for two percent foundation damping. For five percent foundation damping modal analysis gives good agreement (if appropriate modal damping coefficients are chosen) for values of K_1/K_2 of 0.50 and greater.

Considering the case of a type II structure with a rotational foundation the results of the analyses are shown in figure sixty-nine. The response of the type II rotational structure is quite similar to that of the type I rotational structure. The range of validity of modal analysis extends down to a value of K_r/K_1 of about 5000. A two percent foundation damping appears the equivalent of five to ten percent modal damping. The five percent foundation damping relates approximately to between ten and twenty percent modal damping.

For low K_r/K_2 ratios ($K_r/K_2 < 5000$) modal analysis does not seem applicable.

The results of the analog computed and modal computed structural response reduction for a type II structure with a foundation with both rotational and translational freedom is shown in figure seventy. It is observable from the figure that in general shape of the analog response reduction is quite similar to that of the type II rotational structure. Modal analysis appears applicable in the large and intermediate K_1/K_2 range ($K_1/K_2 > 1.5$) with two and five percent foundation damping corresponding to between five and ten and between ten and twenty percent modal damping, respectively. For values of K_1/K_2 less than 1.5 modal analysis no longer has any applicability.

To summarize the detailed information presented in the preceding paragraphs, it has been seen that damping can have a significant effect on structural response. The true effect of damping does not appear, from these limited studies, to be adequately accounted for by use of modal analysis. For large values of damping of ten percent and greater the damping is seen to impart enough resistance to relative motion between the structure and its foundation to result in a structure that can be analytically considered as fixed based. For damping values of below ten percent one must be careful as to the range of foundation stiffness in which the particular structure under investigation is located. If the structure-foundation stiffness ratio is high the use of modal analysis procedures appears valid. However, the effective damping is high and it seems advisable to assign a coefficient of modal damping that is at least five times the foundation damping coefficient. In the region of low K_1/K_2 and K_r/K_2 ratios modal analysis does not appear, even through the assignment of adjusted modal damping coefficients, to have application. Until a more satisfactory method is found for considering damping for structures with low K_1/K_2 and/or K_r/K_2 ratios, it would seem advisable for these structures with soft but moderately to heavily damped foundations ($K_1/K < 0.10$(I), $K_1/K_2 < 1.0$(II), $K_r/K_2 < 2000$, and 5 to 40 percent foundation damping) to assign the structure a fixed base.

L. Modal Representation for Damping

A particularly troublesome aspect of the structure-foundation interaction problem is the finding of a suitable representation for the foundation soil system. The theoretically best approach is to treat the foundation as an elasto-plastic half space. However, only for the very simplest of cases where the foundation is homogenous, elastic and semi-infinite has a solution been determined.

Another approach with merit is to idealize the foundation as a lumped parameter system. Seed and Idriss (39)(40) has recently been quite successful in predicting the response of both homogenous and layered soil systems using such a foundation idealization. This type of representation has the advantage of having the capability of accounting for non-homogenities

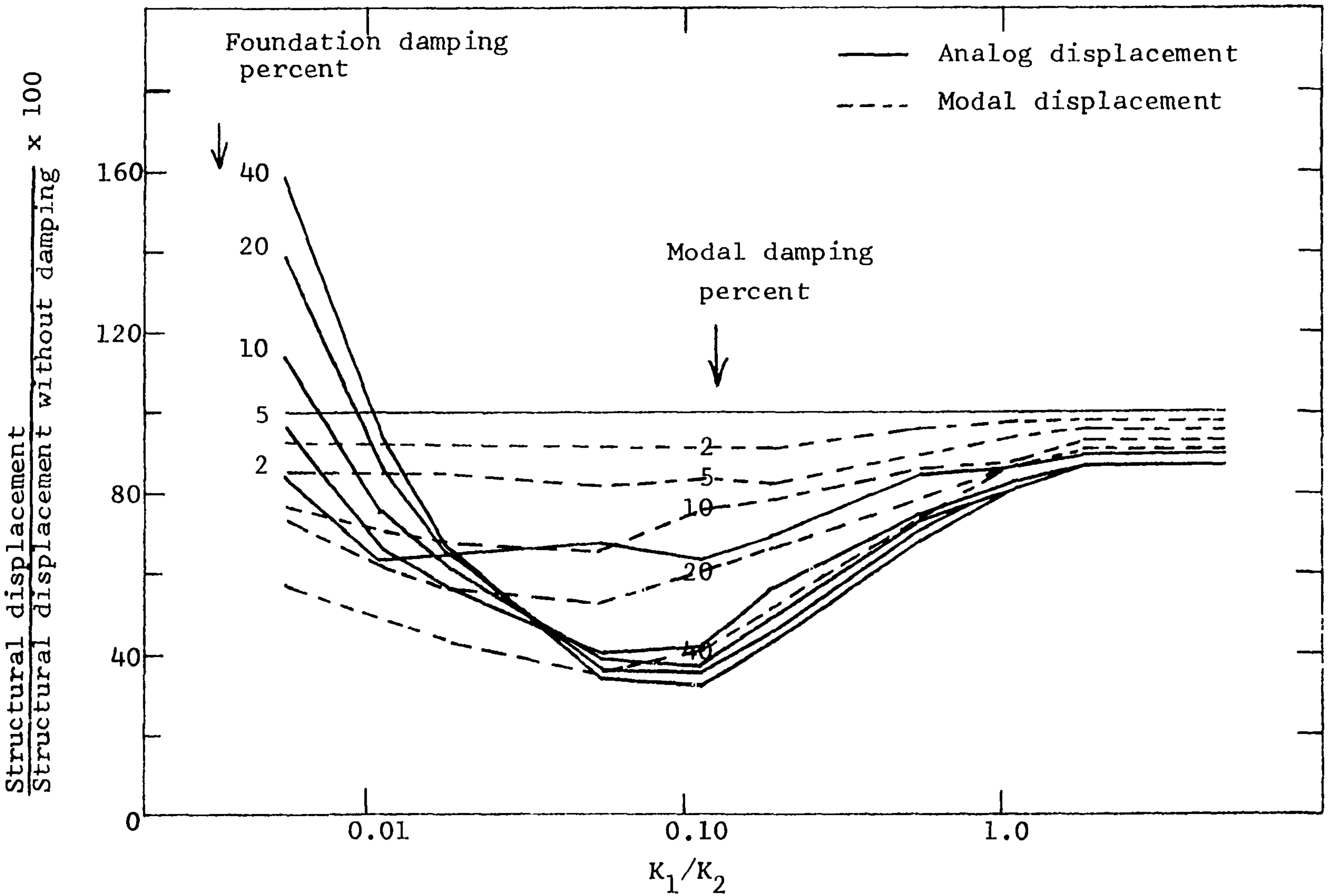


Figure 65. Type I translational structure comparison of analog evaluated and modal absolute sum computed damping effectiveness

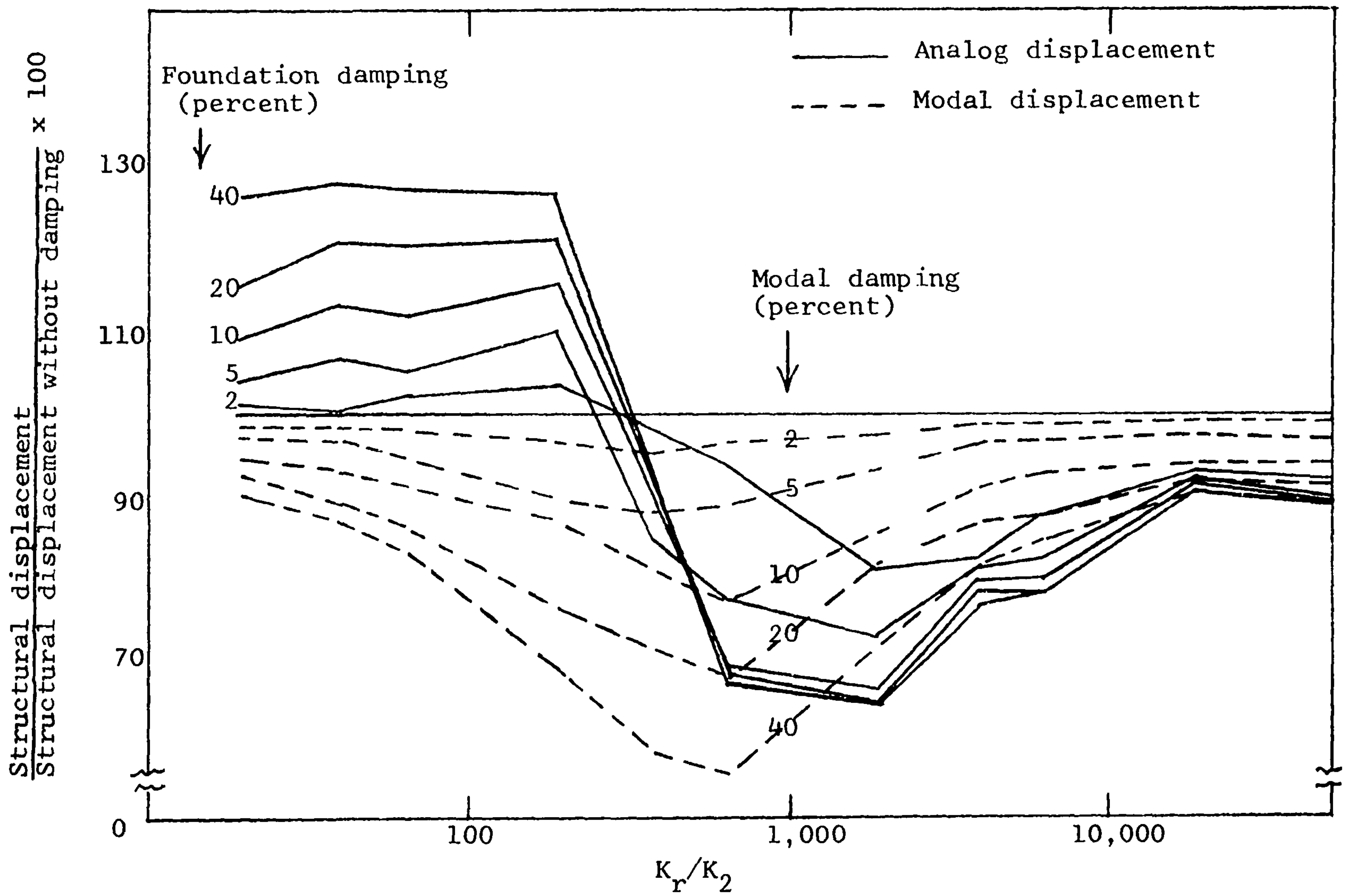


Figure 66. Type I rotational structural comparison of analog evaluated and modal absolute sum computed damping effectiveness

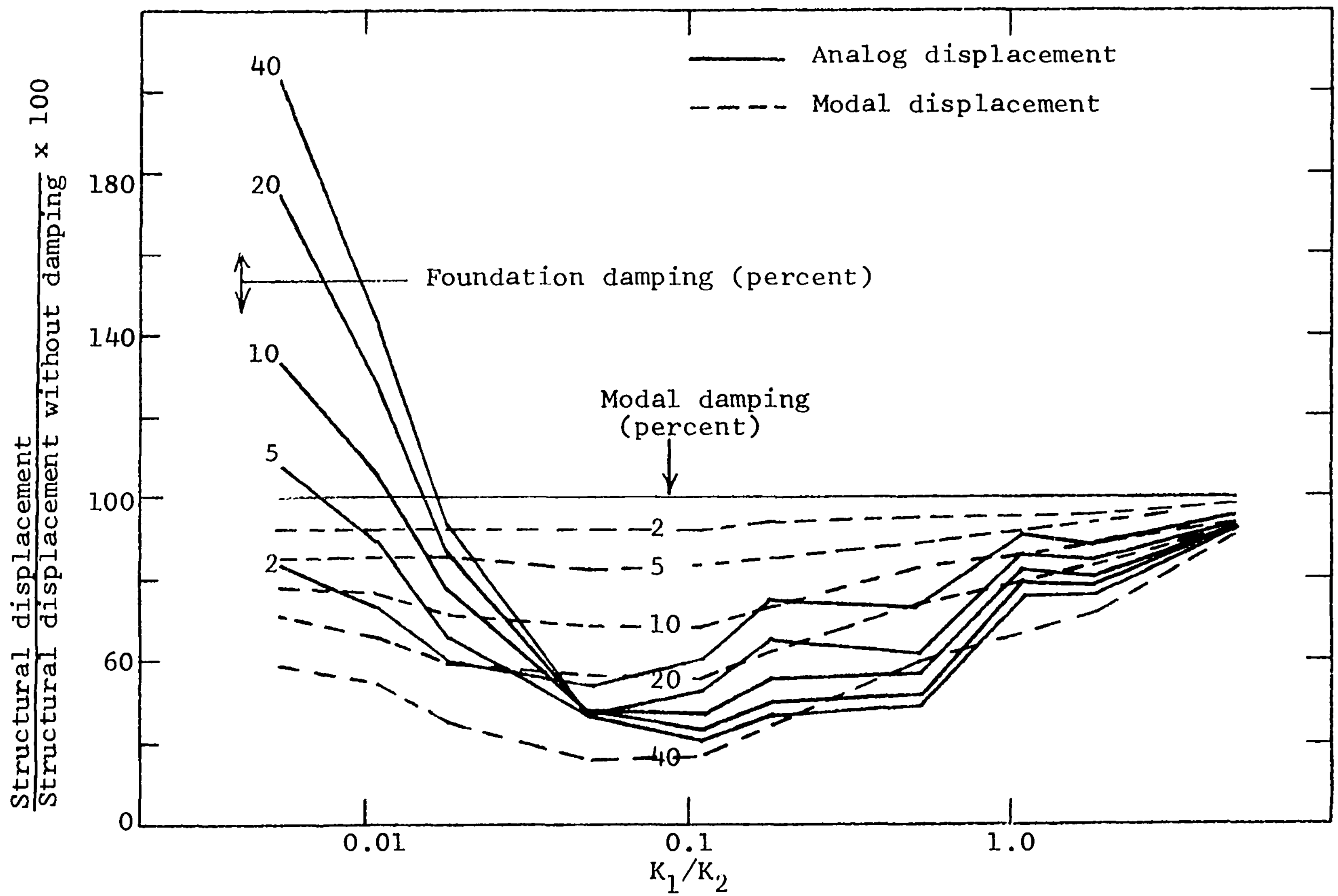


Figure 67. Type I combined variable K_r structure comparison of analog evaluated and modal absolute sum computed damping effectiveness

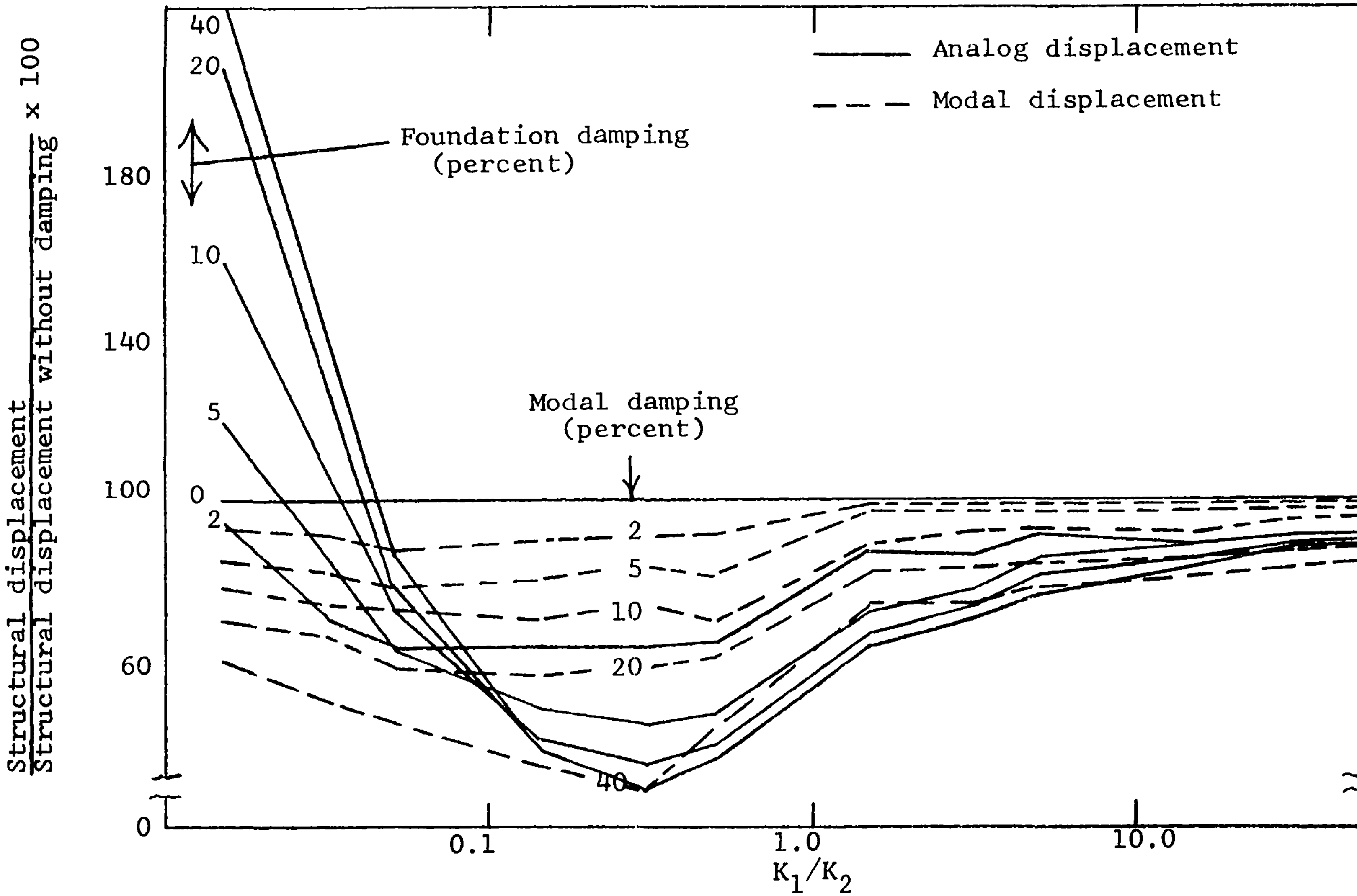


Figure 68. Type II translational structure comparison of analog evaluated and modal absolute sum computed damping effectiveness

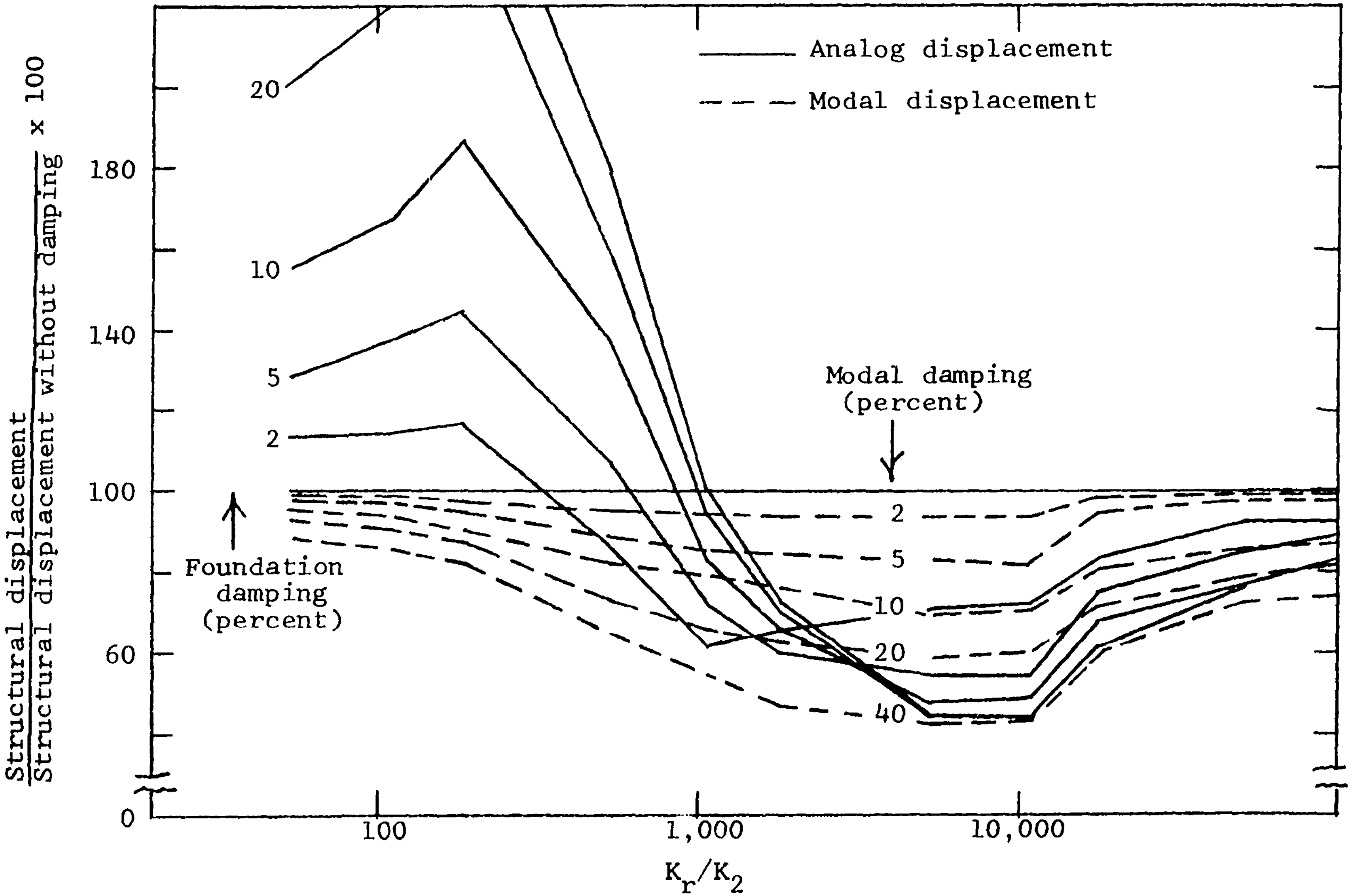


Figure 69. Type II rotational structure comparison of analog evaluated and modal absolute sum computed damping effectiveness

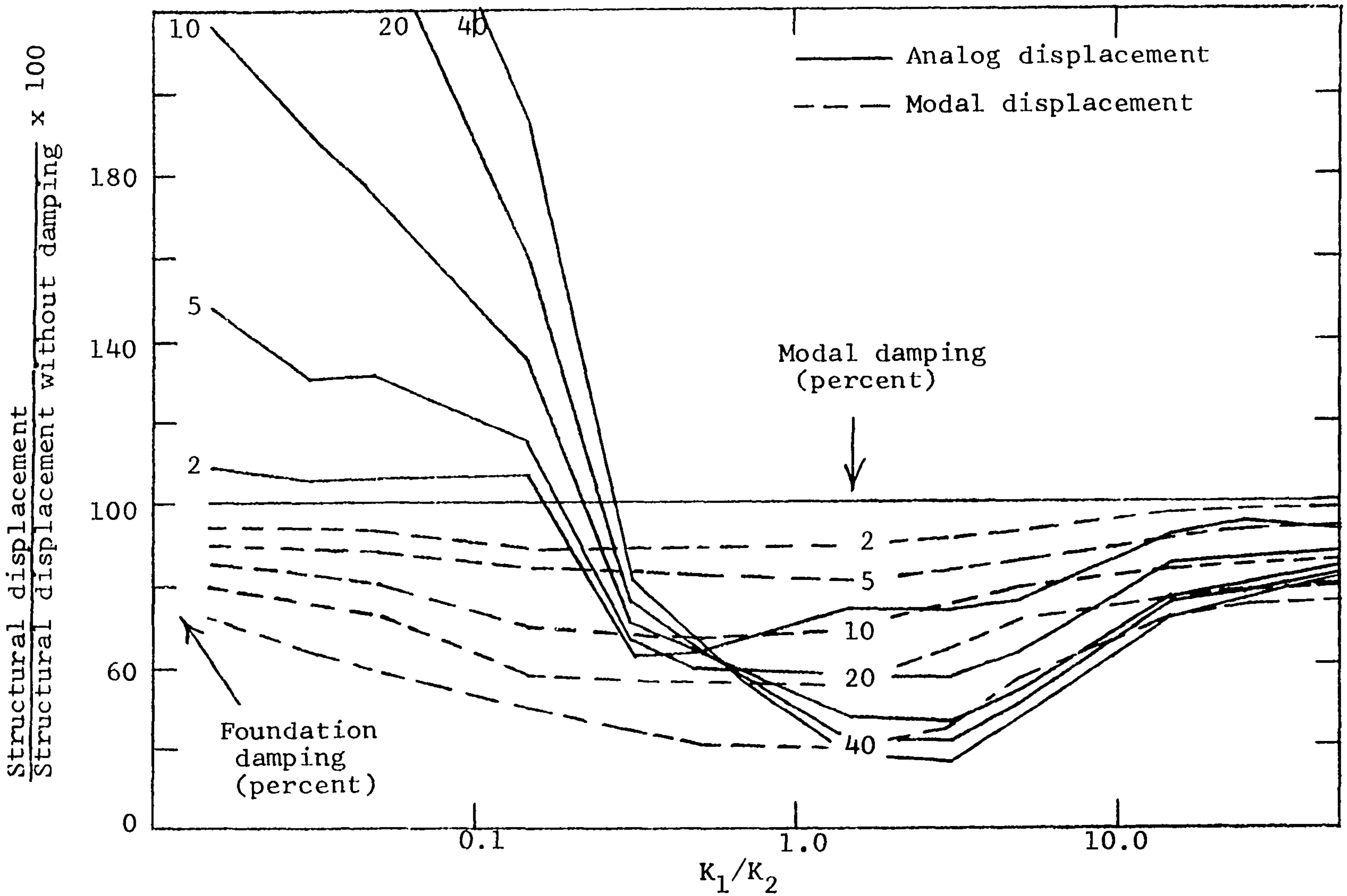


Figure 70. Type II combined variable K_r structure comparison of analog evaluated and modal absolute sum computed damping effectiveness

in soil properties and varying location of bedrock with respect to the structure's base.

The analytical model used in this study is actually the simplest possible lumped parameter foundation idealization, i.e. the foundation is represented by a single lumped mass and a single set of springs and dampers. For a particular site, especially where the foundation is a layered soil system with different stiffness properties for each layer, the use of several masses and stiffnesses in the idealization is desirable. For the purpose of this parametric investigation the characterization of the foundation by single mass and averaged stiffness is undoubtedly suitable, however.

The basic suitability of the structure-foundation model used in this study to properly model foundation damping effects is subject to some question. As indicated previously, damping in the structure-foundation system is of two types, loss of energy in the internal friction of the soil and radiation of energy away from the system. The first type is thought to be reasonably well modeled by the assumption of viscous dampers. However, the second type of damping, the energy loss due to radiation of energy away from the structure, is not rationally accounted for by a viscous damper assumption. It is felt that this effect can, to a closer degree, be identified in a lumped parameter system as the motion imparted to lumped foundation masses by the motion of the structure and base masses through the structure's base. To obtain this effect one must have, as a minimum, a two mass representation of the structure's foundation and underlying soil system. Such a possible idealization is shown in figure seventy-one.

The effect of energy radiation from the structure has, of course, not been accounted for by the model used in this investigation. It is felt that inclusion of this effect by going to a more detailed lumped mass representation of the system could appreciably alter the damping investigation results for low foundation stiffness, high foundation viscosity, cases. As already seen in figures sixty-five through seventy the present model predicts large force transfer into the structure for high foundation viscous damping. This may not be entirely valid for actual structures. It is a topic that certainly merits future study.

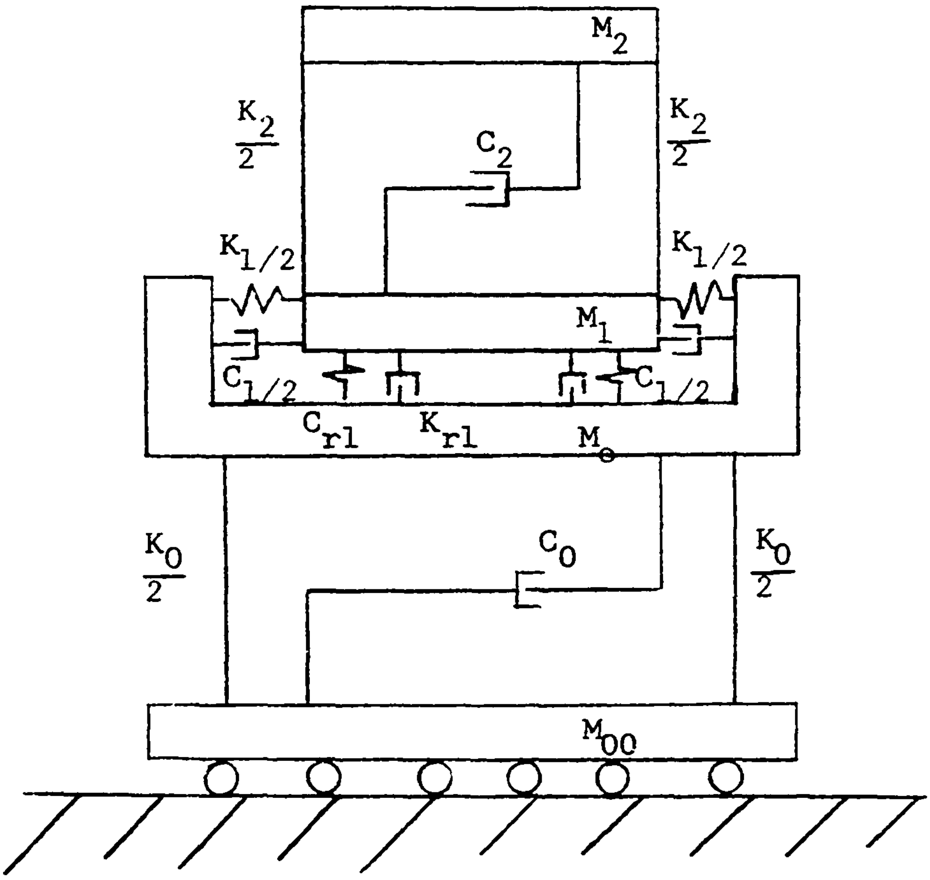


Figure 71. Two foundation mass idealization proposed to account for foundation translational energy radiation from the structure

XI. CONCLUSIONS

The results of this investigation as amplified in the previous discussion section support the following conclusions.

a. In order to achieve accuracy in the analysis of the response of a containment structure to ground motion it is essential that a model appropriate for the particular foundation situation be selected. Restated in another way, it is necessary that both translational and rocking motion be modeled as they present themselves as foundation freedoms. Assumption of different foundation conditions have been shown in this study to yield entirely different structural responses for the same input ground motion.

b. Reduction in foundation stiffness results in reduction in structural modal frequencies for the same structure.

c. The use of the sum of absolute modal maxima for a small number of degrees of freedom (two or three) is supported by the results of this investigation.

d. Containment structures with very firm foundations behave as essentially fixed base structures. As the foundation stiffness is reduced, however, from a very large value to a value approximating the stiffness of the structure a resonance effect occurs and the structural response can easily rise to double the fixed base response. As the foundation stiffness is further reduced, the amount of force actually transmitted into the structure is greatly reduced. The structural response falls to a small fraction (ten to twenty-five percent generally) of the fixed base response at low stiffness ratios.

e. A fairly precise knowledge of foundation stiffness is needed if an accurate determination of structural response to ground motion is to be achieved. A variation in foundation stiffness of a factor of two can result in structural response variations from values that are twice the fixed base response to values of one-half the fixed base response. If only a crude determination of foundation stiffness is made by assignment of stiffness values based solely on a soil classification rather than by actual field studies, for example, the resulting error in stiffness could lead to large over or under estimation of structural response.

f. Change in foundation translational stiffness appears to affect structural response more than does change in rotational stiffness.

g. Based on the small structural response changes that were obtained from rather extreme variations in lumped mass assumption made in this study, the accurate evaluation of structural response appears not to be sensitive to the approximation made in containment structure mass distribution. Any mass distribution that reflects in a reasonable manner the variation in containment mass would probably give accurate structural response results. Specifically, it is thought that for the type I, II, III and IV structures considered in this study, a three or four lump mass system should give good results. For the type V structure five to seven lump mass would probably be required for good results.

h. Even total omission of the mass and/or moment of inertia due to equipment and structures internal to the containment structure has not resulted in a large change in structural response. An estimate of internal structure and equipment mass and moment of inertia to within twenty-five percent should be satisfactory.

i. Structural displacement is much greater in tall containment structures than short ones. More important, however, is the observation that the seismic stress developed in a containment structure is very sensitive to the structure's height. A tall containment structure develops much larger seismic stress than does a short one (for the same containment volume). Thus, a step the structural designer may take, if seismic stress controls the design and needs to be reduced, is to make a height reduction.

j. An accurate knowledge of the damping properties of the particular foundation under consideration is important. Large value of foundation damping (twenty percent and greater) can, in effect, be nearly equivalent dynamically to the assumption of a fixed base system. Even small values of damping (two to five percent) can result in structural response values that are greatly decreased in the resonance region and greatly increased in the low stiffness region over the values of the same system without damping.

k. Modal analysis appears to have only limited application in structural response problems that involve structure-foundation interaction and damping. The results of this study support the use of the modal analysis method only for moderate to large stiffness ratios (K_1/K_2 or K_r/K_2) when the foundation has a small amount (two to five percent) of damping. For the foundation situation where a low stiffness but a moderate to large amount of damping is present, modal analysis predicts low structural response values whereas the results of the analog investigations show the structures actually to behave as essentially fixed based. Where a very soft (low K_1 or K_r as applicable) foundation occurs and where the foundation damping is also small (ten percent and less) modal analysis has also not been successful in evaluating the actual structural action.

l. A small amount of structural damping has not been found effective in limiting structural response. However, a small amount of foundation damping, i.e. foundation damping percentages of two and five percent, have been shown to be equivalent to much larger (ten to forty) percentages of modal damping. Thus, the assignment of five times the evaluated foundation damping percentage as a modal damping percentage (for all modes) seems useful in determining structural response for moderate to stiff, lightly damped, foundations.

XII. SUGGESTIONS FOR FURTHER STUDY

It is probably not too surprising that in the course of this investigation several topics meriting more extended study emerged. Topics in which development of further information appears important to the accurate analysis of seismic forces in containment structures are as follows:

a. The structural response results of this study and the conclusions derived therefrom have been based on a bilinearly approximated El Centro spectrum. This spectrum shows a large response over a broad range of frequency values and is the spectrum most generally used where a conservatively large maximum response is desired. The results of its bilinearly approximated usage in this study and the conclusions derived herein are qualitative, not quantitative, results, however, since only this one approximate spectrum has been used. The results show the effect of foundation conditions in general terms but, of course, could not be used for design since the effect of selected design earthquake spectrum would qualitatively differ somewhat from the bilinear response spectrum used herein. Even if the El Centro spectrum was used as one of the design spectra, it would be desirable to more accurately approximate this spectra than that approximation afforded by the bilinear approximation used in this study. It would, therefore, be desirable to extend this study to ascertain the effect that varying the actual earthquake will have on the structure's response. Such a study might, for example, include use of four or five of the presently most used spectra. Such spectra could fairly easily and accurately be approximated by a series of twenty-five to fifty straight line segments. The results of such information could enable the designer to "envelope" his actual containment response for the particular design K_1/K_2 or K_r/K_2 situation.

b. The assignment of approximate structural stiffness values for containment structures is a problem. For monolithic structures of large length-to-depth ratios (>3.0) the deflection due to shear is neglibible and inclusion of shear deflection is not needed. As this ratio decreases into the range of tall containment structures (1.5 to 3.0) the shear contribution to deflection must be considered. For containment structures that have length-to-depth ratios of one to one and one-half the mechanics of materials plane strain assumption is not valid and reliance must be placed on structural testing and/or more sophisticated analysis. For stubby shear walls of solid rectangular cross section tests have been made and design curves are available. However, no such information could be identified in the literature for hollow circular cross sections. Research directed toward development of stiffness information for such a typical containment shape would be desirable.

c. The containment structure analyst is at the outset faced with making the decision as to the number of lumped masses to use in dynamically describing his structure. A containment structure is, of course, a continuous mass system with a very non-uniform mass distribution. Some guidance may be afforded by comparing the known frequencies and mode shapes of distributed mass cantilever or simply supported beams or other simple cases with the results obtained by using various mass lumping approximations. The extendibility of such information to a containment structure with its varying possible foundation freedoms and stiffnesses is unknown. It would be useful, therefore, to study the structural response of typical containment structures with each of their typical foundations and varied foundation stiffnesses for a series of problems in which only the lumped mass approximation to the structures is varied. The results derived from such a study could serve to give definitive guidance to the structure designer as to how extensive a lumped mass system is necessary to accurately describe a containment structure dynamics problem.

d. The analog computer techniques used in this study are felt to have real possibility as an analysis tool to evaluate the response of structure-foundation systems to seismic ground motion. However, it is considered that at least one more mass with translational freedom must be included in the foundation model in an attempt to account for energy radiation from the structure. For containment structures at least one more mass with far coupled stiffness should also be added to increase the accuracy of the structure mass representation. For more common structures other numbers of masses, as appropriate, should be added and the capability of the analog computer to account for nonlinear structural resistance should also be remembered. The behavior of such a model with appropriate coefficients could logically be subjected to a series of filtered white noise inputs and the results compared with those obtained from actual measurement on a structure due to earthquake motion.

e. The assignment of analog computer damping coefficients has been based on the known damping characteristics of one degree of freedom structures and foundations (see section VI). For the two mass approximation used in this study such a procedure was suitable. However, when additional masses are included to more accurately describe the structure and/or foundation a good general method for assigning damping coefficients is not available. Research directed toward proper characterization of such coefficients in a damped multidegree-of-freedom system is a necessary prerequisite for extending this method into the non-proportional damping, many mass range.

f. The modal analysis technique incorporating the assignment of modal damping coefficients does not appear as a very useful general analytical tool when the results of the analog-digital comparison in this study are considered. Foss (13) has developed a technique to decouple the equations of motion into a system of exponentially damped varying phase angle modes. The setup and coding of his method for digital computation of the seismic problem appears promising. It would be interesting to compare the results of such an analytical study with those presented in the analog portion of this investigation.

g. There appears to be a real potential for increasing the seismic resistance of containment structures by inclusion of a low stiffness layer between the structure's base and its foundation. This investigation has indicated that proper mat selection could conceivably result in a ten-fold increase in containment seismic resistance. It could possibly increase the resistance of the structure to fault motion, as well. The importance of a precise knowledge of the in-place stiffness and damping properties of such a mat cannot be overemphasized. Further study of the possibility of enhancement of containment seismic resistance by use of such mats is suggested.

XIII. LITERATURE CITED

1. Barkan, D. D. Dynamics of bases and foundations. New York, N.Y., McGraw Hill Book Co., Inc. 1962.

2. Belsheim, R. O. and O'Hara, G. J. Shock design of shipboard equipment, Part I - Dynamic design analysis method. United States Naval Research Laboratory Report 5545. 1960.

3. Biggs, J. M. Introduction to structural dynamics. New York, N.Y., McGraw-Hill Book Co., Inc. 1964.

4. Biot, M. A. Analytical and experimental methods in engineering seismology. American Society of Civil Engineers Transactions 108:365-408. 1943.

5. Blume, J. A. and Meehan, J. F. A structural dynamics research program on actual school buildings. Second World Conference on Earthquake Engineering Proceedings 2:1297-1327. 1960.

6. Blume, J. A., Newmark, N. M. and Corning, L. H. Design of multistory reinforced concrete buildings for earthquake motions. Chicago, Ill., Portland Cement Association. 1961.

7. Bycroft, G. N. White noise representation of earthquakes. American Society of Civil Engineers Proceedings 86 No. EM2:1-16. 1960.

8. Clough, R. W. Earthquake analysis by response spectrum superposition. Seismological Society of America Bulletin 52 No. 3:674-660. 1962.

9. Cole, D. G. The damping capacity of hardened cement paste, motar and concrete specimens. In Skipp, B. O., ed. Vibration in engineering. Pp. 235-247. London, Butterworths, Inc. 1966.

10. DaDeppo, D. A. Damping in discrete linear elastic systems. American Society of Civil Engineers Proceedings 89 No. EM2:13-18. 1963.

11. Duke, C. M. Effects of ground on destructiveness of large earthquakes. American Society of Civil Engineers Proceedings 84 No. SM3:1730. 1958.

12. Fleming, J. F., Screwvala, F. N. and Kondner, R. L. Foundation superstructure interaction under earthquake

motion. Third World Conference on Earthquake Engineering Proceedings 1:22-30. 1965.

13. Foss, K. A. Coordinates which uncouple the equations of motion of damped linear systems. American Society of Mechanical Engineers Transactions 80 No. APM:361-364. 1958.

14. Housner, G. W. Behavior of structures during earthquakes. American Society of Civil Engineers Transaction 85 No. EM4:109-129. 1959.

15. Housner, G. W. Interaction of building and ground during an earthquake. Seismological Society of America Bulletin 47 No. 3:179-186. 1957.

16. Housner, G. W. and Brady, A. G. Natural periods of buildings. American Society of Civil Engineers Proceedings 89 No. EM4:31-65. 1963.

17. Housner, G. W., Martel, R. R. and Alford, J. L. Spectrum analysis of strong motion earthquakes. Seismological Society of America Bulletin 43 No. 2:97-119. 1953.

18. Housner, G.W. and McCann, G. D. The analysis of strong motion earthquake records with the electric analog computer. Seismological Society of America Bulletin 39 No. 1:47-56. 1949.

19. Jacobsen, L. S. Natural period of uniform cantilever beam beams. American Society of Civil Engineers Transactions 104:402-439. 1939.

20. Jacobsen, L. S. Steady state forced vibrations as influenced by damping. American Society of Civil Engineers Transactions 52:169. 1930.

21. Jennings, P. C. and Kuroiwa, J. H. Vibration and soil-structure interaction tests of a nine-story reinforced concrete building. Seismological Society of America Bulletin 58 No. 3:891-916. 1968.

22. Kanai, K. and Yoshizawa, S. On the damping of vibration of actual buildings. Earthquake Research Institute of Japan Bulletin 30. 1952.

23. Korn, G. A. and Korn, T. M. Electronic analog computers. 2nd ed. New York, N.Y., McGraw-Hill Book Co., Inc. 1956.

24. Lycan, D. L. and Newmark, N. M. Effect of structure and foundation interaction. American Society of Civil Engineers Proceedings 87 No. EM5:1-31. 1961.

25. Martel, R. R. Effect of foundation on earthquake motion. Civil Engineering 10 No. 1:7-10. Jan. 1940.

26. Merritt, R. G. and Housner, G. W. Effect of foundation compliance on earthquake stresses in multistory buildings. Seismological Society of America Bulletin 44 No. 4:551-569. 1954.

27. Mindlin, R. D., Stubner, F. W. and Cooper, H. L. Response of damped elastic systems to transient disturbances. S Society of Experimental Stress Analysis Proceedings 5 No. 2:69-87. 1948.

28. Murphy, V. A. The effect of ground characteristics on the aseismic design of structures. Second World Conference on Earthquake Engineering Proceedings 1:231-247. 1960.

29. Nakagawa, K. Vibration characteristics of reinforced concrete buildings existing in Japan. Second World Conference on Earthquake Engineering Proceedings 2:973. 1960.

30. O'Hara, G. J. Effect of shock spectra on the dynamic reaction of structures. Society for Experimental Stress Analysis Proceedings 17:145. 1961.

31. O'Hara, G. J. Shock spectra and design shock spectra. United States Naval Research Laboratory Report 5386. 1959.

32. Parmelee, R. A. Building-foundation interaction effects. American Society of Civil Engineers Proceedings 93 No. EM2:131-152.

33. Programmer's Manual. System 360 Scientific Subroutine Package (360A-CM-03X). H20-0205-3. White Plains, N.Y., International Business Machines, Inc. 1968.

34. Rogers, G. L. An introduction to the dynamics of framed structures. New York, N.Y., John Wiley and Sons, Inc. 1959.

35. Rosenblueth, E. Earthquake resistant design. Applied Mechanics Review 14:923. 1961.

36. Salvadori, M. G. and Heer, E. Periods of framed buildings for earthquake analysis. American Society of Civil Engineers Transactions 127 No. 1:660-678. 1962.

37. Scarvazzo, R. J. Foundation-structure interaction in the analysis of wave motions. Seismological Society of America Bulletin 57 No. 4:735-746. 1967.

38. Schenker, L. The dynamic response of tall structures to lateral loads. American Society of Civil Engineers Proceedings 82:1-13. 1956.

39. Seed, H. B. and Idriss, I. M. Influence of ground motions during earthquakes. American Society of Civil Engineers Proceedings 95 No. SM1:99-137. 1969.

40. Seed, H. B. and Idriss, I. M. Seismic response of horizontal soil layers. American Society of Civil Engineers Proceedings 94 No. SM4:1003-1029. 1968.

41. Thomson, W. T. A survey of the coupled ground-building vibrations. Second World Conference on Earthquake Engineering Proceedings 2:833-847. 1960.

42. Thomson, W. T. Vibration theory and applications. Englewood Cliffs, N.J., Prentice-Hall, Inc. 1965.

43. Whitman, R. V. Analysis of foundation vibrations. In Skipp, B. O., ed. Vibration in engineering. Pp. 159-179. London, Butterworths. 1966.

XIV. ACKNOWLEDGEMENTS

The author wishes to express his appreciation to Dr. Ti-Ta Lee for his encouragement and suggestions during this investigation.

Thanks are also due Dr. Carl Ekberg, Dr. Raymond Untrauer, Dr. Richard Danofsky and Dr. Alfred Robach, all of whom were from time to time interrupted from their schedules to be involved in discussions relating to the problem.

Finally, the support of the United States Atomic Energy Commission whose generous leave of absence made possible the program of study during which this investigation was undertaken is gratefully acknowledged.

XV. APPENDIX

A. Computer Programs Developed for and Used in the Analytical Investigation

1. Tabulation of computer program variable identifiers

M	Degrees of freedom of problem
IMZ	Number of mass cases
ISZ	Number of stiffness situations per mass case
VALA	Value of constant coefficient in $R=AX^M$ approximation of response spectrum for frequencies less than 0.25 cps
VALB	Value of exponent in $R=AX^M$ approximation of response spectrum for frequencies less than 0.25 cps
VALC	Value of constant coefficient in $R=AX^M$ approximation of response spectrum for frequencies greater than 0.25 cps
VALD	Value of exponent in $R=AX^M$ approximation of response spectrum for frequencies greater than 0.25 cps
ICTR	Counter on mass cases
A(I,J)	Mass matrix
B(I,J)	Stiffness matrix
W(I,J)	Dummy for storage of stiffness matrix
F(I)	Forcing function matrix
Y(I,J)	Dummy for storage of mass matrix
FRSR	Foundation rotational stiffness ratio (K_N/K_r)
FTSR	Foundation translational stiffness ratio (K_2/K_r)

NROOT	Subroutine (IBM) for computation of solution for matrix equation of the AX= λBX type
XL(I)	Dummy vector used to develop circular frequency squared and frequency values
XNF(I,J)	Matrix of normalization factor values
Q(I,J)	Matrix of normalized eigenvector
P(I)	Matrix of $e^T F$ values
S(I,J)	Matrix of structural modal response values
QT(I,J)	Transpose of Q matrix
SDOF(I)	Spectral response vector
RRM(I,J)	Absolute value of relative structural response vector
FVM(I)	Absolute sum structural response vector
FMS(I)	Square root of sum of squares structural response vector
ARA	Computed fixed base λ^2 value
ARB	Computed fixed base λ value
FBF	Fixed base frequency value
FBR	Fixed base structural response value
X(I,J)	Matrix of eigenvectors

2. Main program

```
      STRUCTURAL SYSTEM RESPONSE BY MODAL ANALYSIS
      DIMENSION A(3,3),B(3,3),XL(3) X(3,3),AA(9),BB(9),XX(9),F(3),
     1QT(3,3),P(3),C(3,3),XT(3,3),XNF(3,3),Q(3,3),D(3,3),S(3,3),
     1R(3),W(3,3),SDOF(3),RRM(3,3),FVM(3),FVS(3),Y(3,3)
      EQUIVALENCE (A(1,1),AA(1)),(B(1,1),BB(1)),(X(1,1),XX(1))
      READ (1,1) M,IMZ,ISZ,VALA,VALB,VALC,VALD
    1 FORMAT (3I5,4F10.5)
      DO 99 L=1,IMZ
      ICTR=0
      DO 6 I=1,M
    6 READ (1,3) (B(I,J),J=1,M)
```

```
      DO 98 I=1,M
      DO 98 J=1,M
   98 W(I,J) = B(I,J)
      READ (1,3) (F(I),I=1,M)
      DO 99 N=1,ISZ
      DO 2 J=1,M
    2 READ (1,3) (A(J,K),K=1,M)
      DO 78 I=1,M
      DO 78 J=1,M
   78 Y(I,J)=A(I,J)
    3 FORMAT (4F20.0)
      WRITE (3,40) L
   40 FORMAT (15H1 STRUCTURE TYPE  I4)
      WRITE (3,7)
    7 FORMAT ( ' MASS MATRIX ' )
      DO 62 I=1,M
      DO 62 J=1,M
   62 B(I,J) = W(I,J)
      DO 8 I=1,M
    8 WRITE (3,9) (B(I,J),J=1,M)
      WRITE (3,13)
   13 FORMAT ( ' FORCING FUNCTION MATRIX ' )
      WRITE (3,21) (F(I),I=1,M)
      WRITE (3,50)
   50 FORMAT ( ' FOUNDATION ROTATIONAL STIFFNESS RATIO ' )
      FRSR=A(M,M)/A(1,1)
      WRITE (3,21) FRSR
      WRITE (3,60)
   60 FORMAT ( ' FOUNDATION TRANSLATION STIFFNESS RATIO ' )
      FTSR=(A(2,2)-A(1,1))/A(1,1)
      WRITE (3,21) FTSR
      WRITE (3,4)
    4 FORMAT ( ' STIFFNESS MATRIX ' )
      DO 5 I=1,M
    5 WRITE (3,9) (A(I,K),K=1,M)
    9 FORMAT (6F21.7)
      DO 30 I=1,M
      DO 30 J=1,M
   30 D(I,J)=B(I,J)
      CALL NROOT (M,AA,BB,XL,XX)
      WRITE (3,10)
   10 FORMAT ( ' EIGENVALUE SQUARED VECTOR ' )
      WRITE (3,21) (XL(I),I=1,M)
      DO 55 I=1,M
      XL(I)=SQRT(XL(I))
   55 XL(I)=XL(I)/6.2800
      WRITE (3,31)
   31 FORMAT ( ' FREQUENCY VECTOR ' )
      WRITE (3,21) (XL(I),I=1,M)
      WRITE (3,11)
   11 FORMAT ( ' EIGENVECTOR MATRIX ' )
```

```
      DO 12 I=1,M
   12 WRITE (3,21) (X(I,J),J=1,M)
   21 FORMAT (10F13.5)
      DO 17 I=1,M
      DO 17 J=1,M
      C(I,J)=0.0
      DO 17 K=1,M
   17 C(I,J)=C(I,J) +D(I,K)*X(K,J)
      DO 27 I=1,M
      DO 27 J=1,M
   27 XT(I,J)=X(J,I)
      DO 18 I=1,M
      DO 18 J=1,M
      XNF(I,J)=0.0
      DO 18 K=1,M
   18 XNF(I,J)=XNF(I,J)+XT(I,K)*C(K,J)
      WRITE (3,19)
   19 FORMAT ( ' NORMALIZATION FACTOR SQUARED MATRIX ' )
      DO 20 I=1,M
   20 WRITE (3,21) (XNF(I,J),J=1,M)
      DO 25 J=1,M
      DO 25 I=1,M
   25 XNF(I,J)=SQRT(XNF(J,J))
      DO 22 I=1,M
      DO 22 J=1,M
   22 Q(I,J)=0.0
      DO 23 J=1,M
      DO 23 I=1,M
   23 Q(I,J)=X(I,J)/XNF(J,J)
      WRITE (3,24)
   24 FORMAT ( ' NORMALIZED EIGENVECTOR MATRIX ' )
      DO 28 I=1,M
   28 WRITE (3,21) (Q(I,J),J=1,M)
      MMOD= M-2
      DO 75 I=1,MMOD
      DO 75 J=1,M
      S(I,J) = 0.0
   75 S(I,J)=Q(I,J)-Q(I+1,J)
      WRITE (3,76)
   76 FORMAT ( ' STRUCTURAL RESPONSE MATRIX ' )
      DO 77 I=1,MMOD
   77 WRITE (3,21) (S(I,J),J=1,M)
      DO 14 I=1,M
      DO 14 J=1,M
   14 QT(I,J)=Q(J,I)
      DO 15 I=1,M
      P(I)=0.0
      DO 15 J=1,M
   15 P(I)= P(I) + QT(I,J)*F(J)
      WRITE (3,16)
   16 FORMAT ( ' PARTICIPATION FACTORS ' )
```

```
      WRITE (3,21) (P(I),I=1,M)
      DO 81 I=1,MMOD
      DO 81 J=1,M
      R(I,J) = 0.0
   81 R(I,J) = S(I,J)*P(J)
      WRITE (3,82)
   82 FORMAT ( ' INTERMEDIATE ANSWER MATRIX ' )
      DO 97 I=1,MMOD
   97 WRITE (3,21) (R(I,J),J=1,M)
      SUBSECTION FOR DETERMINATION OF SPECTRAL RESPONSE
      DO 41 I=1,M
      IF(XL(I)-0.2500) 42,42,43
   42 SDOF(I)=VALA*XL(I)**VALB
      GO TO 41
   43 SDOF(I)=VALC*XL(I)**VALD
   41 CONTINUE
      SUBSECTION FOR DETERMINATION OF MAXIMUM POSSIBLE
      RELATIVE MASS DISPLACEMENT
      DO 44 I=1,MMOD
      DO 44 J=1,M
   44 RRM(I,J)=SDOF(J)*R(I,J)
      DO 45 I=1,MMOD
      DO 45 J=1,M
   45 RRM(I,J)=ABS(RRM(I,J))
      DO 46 I=1,MMOD
      FVM(I)=0.0
      DO 46 J=1,M
   46 FVM(I)=FVM(I)+RRM(I,J)
      WRITE (3,47)
   47 FORMAT ( ' ABSOLUTE RESPONSE ' )
      DO 58 I=1,MMOD
   58 WRITE (3,21) FVM(I)
      SUBSECTION FOR DETERMINATION OF SUM OF SQURES
      RELATIVE MASS DISPLACEMENT
      DO 48 I=1,MMOD
      FVS(I)+0.0
      DO 48 J=1,M
      RRM(I,J)=RRM(I,J)*RRM(I,J)
   48 FVS(I)=FVS(I)+RRM(I,J)
      DO 49 I=1, MMOD
   49 FVS(I)=SQRT(FVS(I))
      WRITE (3,51)
   51 FORMAT ( ' SUM OF SQUARES RESPONSE ' )
      DO 91 I=1,MMOD
   91 WRITE (3,21) FVS(I)
      SUBSECTION FOR DETERMINATION OF FIXED BASE STRUCTURAL RE-
      WRITE (3,71)                                          SPONSE
   71 FORMAT ( ' RESPONSE SPECTRUM SPECIFICATION DATA ' )
      WRITE (3,21) VALA,VALB,VALC,VALD
      ARA=Y(1,1)/W(1,1)
      ARB=SQRT(ARA)
```

```
      FBF=ARB/6.28
      IF(FBF-0.2500)83,83,84
   83 FBR=VALA*FBF**VALB
      GO TO 85
   84 FBR=VALC*FBF**VALD
   85 CONTINUE
      WRITE (3,86) FBR
   86 FORMAT (33H RESPONSE NORMALIZATION FACTOR IS F8.4)
      SUBSECTION FOR NORMALIZATION OF RESPONSES
      DO 101 I=1,MMOD
      FVM(I)=FVM(I)/FBR
  101 FVS(I)=FVS(I)/FBR
      WRITE (3,102)
  102 FORMAT ( ' NORMALIZED ABSOLUTE RESPONSE ' )
      DO 103 I=1,MMOD
  103 WRITE (3,21) FVM(I)
      WRITE (3,104)
  104 FORMAT ( ' NORMALIZED SUM OF SQUARES RESPONSE ' )
      DO 99 I=1,MMOD
   99 WRITE (3,21) FVS(I)
      STOP
      END
```

3. Subroutine for eigenvalues and eigenvectors of real non-symmetric matrix (31)

```
      COMPUTE EIGENVALUES AND EIGENVECTORS OF B

      K=1
      DO 100 J-2,M
      L=M*(J-1)
      DO 100 I=1,j
      L=L+1
      K=K+1
  100 B(K)=B(L)
          THE MATRIX B IS A REAL SYMMETRIC MATRIX.

      MV=0
      CALL EIGEN (B,X,M,MV)

      FORM RECIPROCALS OF SQUARE ROOT OF EIGENVALUES.  THE RE-
      SULTS ARE PREMULTIPLIED BY THE ASSOCIATED EIGENVECTORS.

      L=0
      DO 110 J=1,M
      L=L+J
  110 XL(J)=1.0/ SQRT( ABS(BL)))
      K=0
      DO 115 J=1,M
```

```
      DO 115 I=1,M
      K=K+1
  115 B(K)=X(K)*XL(J)
      FORM (B**(-1/2))PRIME * A * (B**(-1/2))

      DO 120 I=1,M
      N2=0
      DO 120 J=1,M
      N1=M*(I-1)
      L=M*(J-1)+I
      X(L)=0.0
      DO 120 K=1,M
      N1=N1+1
      N2=N2+1
  120 X(L)=X(L)+B(N1)*A(N2)
      L=0
      DO 130 J=1,M
      DO 130 I=1,j
      N1=I-M
      N2=M*(J-1)
      L=L+1
      A(L)=0.0
      DO 130 K=1,M
      N1=N1+M
      N2=N2+1
  130 A(L)=A(L)+X(N1)*B(N2)

      COMPUTE EIGENVALUES AND EIGENVECTORS OF A

      CALL EIGEN (A,X,M,MV)
      L=0
      DO 140 I=1,M
      L=L+I
  140 XL(I)=A(L)

      COMPUTE THE NORMALIZED EIGENVECTORS

      DO 150 I=1,M
      N2=0
      DO 150 J=1,M
      N1=I-M
      L=M*(J-1)+I
      A(L)=0.0
      DO 150 K=1,M
      N1=N1+M
      N2=N2+1
  150 A(L)=A(L)+B(N1)*X(N2)
      L=0
      K=0
      DO 180 J=1,M
      SUMV=0.0
```

```
      DO 170 I=1,M
      L=L+1
  170 SUMV=SUMV+A(L)*A(L)
  175 SUMV= SQRT(SUMV)
      DO 180 I=1,M
      K=K+1
  180 X(K)=A(K)/SUMV
      RETURN
      END
```

4. Subroutine for eigenvalues and eigenvectors of a real symmetric matrix by Jacobi rotations (31)

```
         SUBROUTINE EIGEN

         SUBROUTINE EIGEN (A,R,N,MV)
         DIMENSION A(1),R(1)

         GENERATE IDENTITY MATRIX

    5 RANGE=1.0E-6
      IF(MV-1) 10,25,10
   10 IQ=-N
      DO 20 J=1,N
      IQ=IQ+N
      DO 20 I=1,N
      IJ=IQ+I
      R(IJ)=0.0
      IF(I-J) 20,15,20
   15 R(IJ)=1.0
   20 CONTINUE

         COMPUTE INITIAL AND FINAL NORMS (ANORM AND ANORMX)

   25 ANORM=0.0
      DO 35 I=1,N
      DO 35 J=1,N
      IF(I-J) 30,35,30
   30 IA=1+(J*J)/2
      ANORM=ANORM+A(IA)*A(IA)
   35 CONTINUE
      IF(ANORM) 165,165,40
   40 ANORM=1.414*SQRT(ANORM)
      ANRMX=ANORM*RANGE/FLOAT(N)

         INITIALIZE INDICATORS AND COMPUTE THRESHOLD, THR

      IND=0
      THR=ANORM
```

```
 45 THR=THR/FLOAT(N)
 50 L=1
 55 M=L+1

       COMPUTE SIN AND COS

 60 MQ=(M*M-M)/2
    LQ=(L*L)/2
    LM=L+MQ
 62 IF( ABS(A(LM))-THR) 130,65,65
 65 IND=1
    LL=L+LQ
    MM=M+MQ
    X=0.5*(A(LL)-A(MM))
 68 Y=-A(LM)/ SQRT(A(LM)*A(LM)+X*X)
    IF(X) 70,75,75
 70 Y=-Y
 75 SINX=Y/ SQRT(2.0*1.0+( SQRT(1.0-Y*Y))))
    SINX2=SINX*SINX
 78 COSX= SQRT(1.0-SINX2)
    COSX2=COSX*COSX
    SINOS = SINX*COSX

       ROTATE L AND M COLUMNS

    ILQ=N*(L-1)
    IMQ=N*(M-1)
    DO 125 I=1,N
    IQ=(I*I-I)/2
    IF(I-L) 80,115,80
 80 IF(I-M) 85,115,90
 85 IM=I+MQ
    GO TO 95
 90 IM=M+IQ
 95 IF(I-L) 100,105,105
100 IL=I+LQ
    GO TO 110
105 IL=L+IQ
110 X=A(IL)*COSX-A(IM)*SINX
    A(IM)=A(IL)*SINX+A(IM)*COSX
    A(IL)=X
115 IF(MV-1) 120,125,120
120 ILR=ILQ+I
    IMR=IMQ+I
    X=R(ILR)*COSX-R(IMR)*SINX
    R(IMR)=R(ILR)*SINX+R(IMR)*COSX
    R(ILR)=X
125 CONTINUE
    X=2.0*A(LM)*SINCS
    Y=A(LL)*COSX2+A(MM)*SINX2-X
    X=A(LL)*SINX2+A(MM)*COSX2+X
```

```
      A(LM)=(A(LL)-A(MM)*SINCS+A(LM)*(COSX2-SINX2)
      A(LL)=Y
      A(MM)=X

         TESTS FOR COMPLETION

         TEST FOR M = LAST COLUMN

  130 IF(M-N) 135,140,135
  135 M=M+1
      GO TO 60

         TEST FOR L = SECOND FROM LAST COLUMN

  140 IF(L-(N-1) 145,150,145
  145 L=L+1
      GO TO 55
  150 IF(IND-1) 160,155,160
  155 IND=0
      GO TO 50

         COMPARE THRESHOLD WITH FINAL NORM

  160 IF(THR-ANRMX) 165,165,45

         SORT EIGENVALUES AND EIGENVECTORS

  165 IQ=-N
      DO 185 I=1,N
      IQ=IQ+N
      LL=I+(I*I)/2
      JQ=N*(I-2)
      DO 185 J=1,N
      JQ=JQ+N
      MM=J+(J*J-J)/2
      IF(A(LL)-A(MM)) 170,185,185
  170 X=A(LL)
      A(LL)=A(MM)
      A(MM)=X
      IF(MV-1) 175,185,175
  175 DO 180 K=1,N
      ILR=IQ+K
      IMR=JQ+K
      X=R(ILR)
      R(ILR)=R(IMR)
  180 R(IMR)=X
  185 CONTINUE
      RETURN
      END
```

5. Typical output data from computer program

```
STRUCTURE TYPE   1
MASS MATRIX
          591.0000000          0.0000000            76900.0000000
            0.0000000       1880.0000000                0.0000000
        76900.0000000          0.0000000         25000000.0000000
FORCING FUNCTION MATRIX
          591.00000         1880.00000             76900.00000
FOUNDATION ROTATIONAL STIFFNESS RATIO
         6329.10900
FOUNDATION TRANSLATIONAL STIFFNESS RATIO
            1.78481
STIFFNESS MATRIX
      1580000.0000000  -1580000.0000000                0.0000000
     -1580000.0000000   4400000.0000000                0.0000000
            0.0000000         0.0000000      99999998000.0000000
EIGENVALUE SQUARED VECTOR
         5820.60100         1287.35500               356.92480
FREQUENCY VECTOR
           12.14854            5.71333                 3.00835
EIGENVECTOR MATRIX
            0.97205            0.78160                 0.92051
           -0.23474            0.62377                 0.39003
           -0.00321           -0.00349                 0.02346
NORMALIZATION FACTOR SQUARED MATRIX
          439.72970            0.00040                 0.00000
            0.00052          977.39230                -0.00195
            0.00073           -0.00049             17870.37000
NORMALIZED EIGENVECTOR MATRIX
            0.04636            0.02500                 0.00689
           -0.01119            0.01995                 0.00292
           -0.00015           -0.00011                 0.00018
STRUCTURAL RESPONSE MATRIX
            0.05755            0.00505                 0.00397
PARTICIPATION FACTORS
           -5.42338           43.70603                23.05142
INTERMEDIATE ANSWER MATRIX
           -0.31211            0.22064                 0.69147
ABSOLUTE RESPONSE
            0.02735
SUM OF SQUARES RESPONSE
            0.01615
RESPONSE SPECTRUM SPECIFICATION DATA
            2.41000            0.13450     0.39000     -1.17500
RESPONSE NORMALIZATION FACTOR IS 0.0328
NORMALIZED ABSOLUTE RESPONSE
            0.83514
NORMALIZED SUM OF SQUARES RESPONSE
            0.49304
```

B. Tabulated Data Used in Analog and Digital Computer Analyses

Table 2. Fixed foundation for parameters for containment structures[1]

Parameter	Structural type I	II	III	IV	V
K_1 (Soft)	$2.82x10^4$	$2.82x10^4$	$2.82x10^4$	$2.82x10^4$	$4.9\ x10^4$
K_1 (Firm)	$2.82x10^5$	$2.82x10^5$	$2.82x10^5$	$2.82x10^5$	$4.9\ x10^5$
K_1 (Stiff)	$2.82x10^6$	$2.82x10^6$	$2.82x10^6$	$2.82x10^6$	$4.9\ x10^6$
K_r (Soft)	$1.00x10^8$	$1.00x10^8$	$1.00x10^8$	$1.00x10^8$	$3.00x10^8$
K_r (Firm	$1.00x10^9$	$1.00x10^9$	$1.00x10^9$	$1.00x10^9$	$3.00x10^9$
K_r (Stiff)	$1.00x10^{10}$	$1.00x10^{10}$	$1.00x10^{10}$	$1.00x10^{10}$	$3.00x10^{10}$
$M_2H_2^{\ 2}$	$10.0\ x10^6$	$47.6\ x10^6$	$10.5\ x10^6$	$54.2\ x10^6$	
$M_2H_2^{\ 2} + I_0$	$25.0\ x10^6$	$62.6\ x10^6$	$25.5\ x10^6$	$69.2\ x10^6$	$21.0\ x10^6$

[1]The values for mass (M_1 and M_2), structural height (H_2) and internal moment of inertia were taken from typical containment data. The values of stiffness and damping were computed using the procedures described in section VI.

Table 3. Structural parameters for two free mass containment structural idealization

Parameter	Structural type I	II	III	IV	V
M_1	1880	2450	1890	2680	2793
M_2	591	763	621	867	1070
H_2	130	250	130	250	140
$M_2H_2{}^2$	10×10^6	47.6×10^6	10.5×10^6	54.2×10^6	21.0×10^6
I_T	25.0×10^6	62.6×10^6	25.5×10^6	69.2×10^6	21.0×10^6
K_2	1.58×10^6	$.56 \times 10^6$	2.16×10^6	$.70 \times 10^6$	1.98×10^6
C_2 (2%)	1.23×10^3	$.83 \times 10^3$	1.46×10^3	1.00×10^3	1.84×10^3
C_2 (5%)	3.07×10^3	2.07×10^3	3.66×10^3	2.46×10^3	4.60×10^3

Table 4. Constants for two free mass analysis of containment structural systems

Case No.	C_2/M_2	K_2/M_2	H_2	C_2/M_1	K_2/M_1	C_1/M_1	K_1/M_1	$C_r/M_2H_2^2+I_0$	$K_r/M_2H_2^2+I_0$	$M_2H_2^2/M_2H_2^2+I_0$
1		2670	130		0841		.0015		004.0	.0031
2	2.08	2670	130	0.654	0841	00.383	.0015	00.40	004.0	.0031
3	2.08	2670	130	0.654	0841	.00766	.0015	00.80	004.0	.0031
4	2.08	2670	130	0.654	0841	.01532	.0015	01.60	004.0	.0031
5		2670	130		0841		.0150		040.0	.0031
6	2.08	2670	130	0.654	0841	.01230	.0150	01.26	040.0	.0031
7	2.08	2670	130	0.654	0841	.02460	.0150	02.52	040.0	.0031
8	2.08	2670	130	0.654	0841	.04900	.0150	05.04	040.0	.0031
9		2670	130		0841		.1500		400.0	.0031
10	2.08	2670	130	0.654	0841	.03830	.1500	04.0	400.0	.0031
11	2.08	2670	130	0.654	0841	.07660	.1500	08.0	400.0	.0031
12	2.08	2670	130	0.654	0841	.15320	.1500	12.0	400.0	.0031
13		0734	250		0162		.0008		1.6	.0030
14	1.09	0734	250	0.240	0162	.00344	.0008	00.254	1.6	.0030
15	1.09	0734	250	0.240	0162	.00568	.0008	00.508	1.6	.0030
16	1.09	0734	250	0.240	0162	.01136	.0008	01.016	1.6	.0030
17		0734	250		0162		.0082		16.0	.0030
18	1.09	0734	250	0 240	0162	.00882	.0082	00.798	16.0	.0030
19	1.09	0734	250	0.240	0162	.01764	.0082	01.600	16.0	.0030
20	1.09	0734	250	0.240	0162	.03528	.0082	03.200	16.0	.0030
21		0734	250		0162		.0820		160.0	.0030
22	1.09	0734	250	0.240	0162	.02840	.0820	02.540	160.0	.0030
23	1.09	0734	250	0.240	0162	.04680	.0820	05.080	160.0	.0030
24	1.09	0734	250	0.240	0162	.11720	.0820	10.160	160.0	.0030
25		3480	130		1140		.0015		4.0	.0030
26	5.89	3480	130	1.94	1140	.00386	.0015	00.40	4.0	.0030
27	5.89	3480	130	1.94	1140	.00772	.0015	00.80	4.0	.0030
28	5.89	3480	130	1.94	1140	.01544	.0015	01.60	4.0	.0030
29		3480	130		1140		.0015		40.0	.0030

Table 4. (Continued)

Case No.	C_2/M_2	K_2/M_2	H_2	C_2/M_1	K_2/M_1	C_1/M_1	K_1/M_1	$C_r/M_2H_2^2+I_0$	$K_r/M_2H_2^2+I_0$	$M_2H_2^2/M_2H_2^2+I_0$
30	5.89	3480	130	1.94	1140	.0243	.0150	01.26	40.0	.0030
31	5.89	3480	130	1.94	1140	.0486	.0150	02.52	40.0	.0030
32	5.89	3480	130	1.94	1140	.0926	.0150	05.04	40.0	.0030
33		3480	130		1140		.1500		400.0	.0030
34	5.89	3480	130	1.94	1140	.0386	.1500	04.0	400.0	.0030
35	5.89	3480	130	1.94	1140	.0772	.1500	08.0	400.0	.0030
36	5.89	3480	130	1.94	1140	.1544	.1500	12.0	400.0	.0030
37		1120	250		0262		.0010		1.45	.0030
38	2.84	1120	250	0.918	0262	.00325	.0010	00.234	1.45	.0030
39	2.84	1120	250	0.918	0262	.00650	.0010	00.468	1.45	.0030
40	2.84	1120	250	0.918	0262	.0130	.0010	00.936	1.45	.0030
41		1120	250		0262		.0105		14.5	.0030
42	2.84	1120	250	0.918	0262	.01020	.0105	00.760	14.5	.0030
43	2.84	1120	250	0.918	0262	.02040	.0105	01.52	14.5	.0030
44	2.84	1120	250	0.918	0262	.04080	.0105	03.04	14.5	.0030
45		1120	250		0262		.1050		145.0	.0030
46	2.84	1120	250	0.918	0262	.0325	.1050	02.4	145.0	.0030
47	2.84	1120	250	0.918	0262	.0650	.1050	04.8	145.0	.0030
48	2.84	1120	250	0.918	0262	.1300	.1050	09.6	145.0	.0030
49		2280	140		0709		.0018		14.3	.0071
50	4.26	2280	140	4.26	0709	.00419	.0018	00.747	14.3	.0071
51	4.26	2280	140	4.26	0709	.0082	.0018	01.494	14.3	.0071
52	4.26	2280	140	4.26	0709	.0167	.0018	03.000	14.3	.0071
53		2280	140		0709		.0175		143.0	.0071
54	4.26	2280	140	4.26	0709	.0133	.0175	02.40	143.0	.0071
55	4.26	2280	140	4.26	0709	.0266	.0175	04.80	143.0	.0071

Table 4. (Continued)

Case No.	C_2/M_2	K_2/M_2	H_2	C_2/M_1	K_2/M_1	C_1/M_1	K_1/M_1	$C_r/M_2H_2^2+I_0$	$K_r/M_2H_2^2+I_0$	$M_2H_2/M_2H_2^2+I_0$
56	4.26	2280	140	4.26	0709	.0526	.0175	09.60	143.0	.0071
57		2280	140		0709		.1750		1430	.0071
58	4.26	2280	140	4.26	0709	.0419	.1750	07.57	1430	.0071
59	4.26	2280	140	4.26	0709	.0838	.1750	15.14	1430	.0071
60	4.26	2280	140	4.26	0709	.1676	.1750	30.28	1430	.0071

Table 5. Constants for two free mass analyses of fixed base containment systems

Case Number	C_2/M_2	K_2/M_2	H_2
1	0	2670	130
2	2.08	2670	130
3	5.20	2670	130
4	0	0734	250
5	1.09	0734	250
6	2.72	0734	250
7	0	3480	130
8	2.36	3480	130
9	5.89	3480	130
10	0	1120	250
11	1.14	1120	250
12	2.84	1120	250
13	0	2280	140
14	1.71	2280	140
15	4.26	2280	140

Table 6. Constants for two free mass analyses of type one containment systems

Case No.	C_2/M_2	K_2/M_2	H_2	C_2/M_1	K_2/M_1	C_1/M_1	K_1/M_1	C_r/I_T	K_r/I_T
16	2.08	2670	130	0.654	0841	00.45	0015	00.20	004.0
17	2.08	2670	130	0.654	0841	00.90	0015	00.40	004.0
18	2.08	2670	130	0.654	0841	01.80	0015	00.80	004.0
19	2.08	2670	130	0.654	0841	03.60	0015	01.60	004.0
20	2.08	2670	130	0.654	0841	01.42	0150	00.63	040.0
21	2.08	2670	130	0.654	0841	02.85	0150	01.26	040.0
22	2.08	2670	130	0.654	0841	05.70	0150	02.52	040.0
23	2.08	2670	130	0.654	0841	11.40	0150	05.04	040.0
24	2.08	2670	130	0.654	0841	04.50	1500	01.50	400.0
25	2.08	2670	130	0.654	0841	09.00	1500	03.00	400.0
26	2.08	2670	130	0.654	0841	18.00	1500	06.00	400.0
27	2.08	2670	130	0.654	0841	36.00	1500	12.00	400.0

Table 7. Tabulation of selected ratios, type I structure

No base translation			No base rotation		
K_r	K_r/K_2	$K_r/MH_2^2+I_0$	K_1	K_1/K_2	K_1/M_1
$30.0\text{x}10^6$	19.0	001.2	$.845\text{x}10^4$	0.00535	00004.5
$60.0\text{x}10^6$	38.0	002.4	$1.69\ \text{x}10^4$	0.01070	00009.0
$100.0\text{x}10^6$*	63.3	004.0*	$2.82\ \text{x}10^4$	0.01790	00015. *
$300.0\text{x}10^6$	190.0	012.0	$8.45\ \text{x}10^4$	0.05350	00045.
$600.0\text{x}10^6$	380.0	024.0	$1.69\ \text{x}10^5$	0.1070	00090.
$1.0\text{x}10^9$*	633.0	040.0*	$2.82\ \text{x}10^5$	0.1790	00150. *
$3.0\text{x}10^9$	1900.0	120.0	$8.45\ \text{x}10^5$	0.535	00450.
$6.0\text{x}10^9$	3800.0	240.0	$1.69\ \text{x}10^6$	1.070	00900.
$10.0\text{x}10^9$*	6330.0	400.0*	$2.82\ \text{x}10^6$	1.790	01500. *
$30.0\text{x}10^9$	19000.0	1200.0	$8.45\ \text{x}10^6$	5.35	04500.
$60.0\text{x}10^9$	38000.0	2400.0	$1.69\ \text{x}10^7$	10.70	09000.
$100.0\text{x}10^9$	63000.0	4000.0	$2.82\ \text{x}10^7$	17.90	15000.

*Note $K_2 = 1.58 \times 10^6$

$M_2H_2^2+I_0 = 25.0 \times 10^6$

$M_1 = 1880.$

Translation/Rotation				
K_1	K_1/K_2	K_r	K_1/M_1	$K_r/M_2H_2^{\,2}+I_0$
$0.845\text{x}10^4$	.00535	$30.0\text{x}10^6$	000045.	0001.2
$1.69\ \text{x}10^4$	.01070	$60.0\text{x}10^6$	0000.9	0002.4
$2.82\ \text{x}10^4$	.01790	$100.0\text{x}10^6$	0001.5	0004.0
$8.450\text{x}10^4$	.0535	$300.0\text{x}10^6$	0004.5	0012.0
$1.69\ \text{x}10^5$	.1070	$600.0\text{x}10^6$	0009.0	0024.0
$2.82\ \text{x}10^5$	.1790	$10\ \ \text{x}10^9$	0015.0	0040.0
$8.45\ \text{x}10^5$	.535	$3.0\text{x}10^9$	0045.0	0120.
$1.69\ \text{x}10^6$	1.070	$6.0\text{x}10^9$	0090.0	0240.
$2.82\ \text{x}10^6$	1.790	$10.0\text{x}10^9$	0150.0	0400.
$8.45\ \text{x}10^6$	5.35	$30.0\text{x}10^9$	0450.0	1200.
$1.69\ \text{x}10^7$	10.70	$60.0\text{x}10^9$	0900.0	2400.
$2.82\ \text{x}10^7$	17.90	$100.0\text{x}10^9$	1500.0	4000.0

Table 8. Tabulation of constants for type I structure two free mass analysis, no damping

Case Number	K_2/M_2	H_2	M_2H_2/I_T	K_r/I_T	K_2/M_1	K_1/M_1
I 1	2670.	130.	0.0031	001.2	0841.	0000.45
I 2	2670.	130.	0.0031	002.4	0841.	0000.90
I 3*	2670.	130.	0.0031	004.0	0841.	0001.5
I 4	2670.	130.	0.0031	012.0	0841.	0004.5
I 5	2670.	130.	0.0031	024.0	0841.	0009.0
I 6*	2670.	130.	0.0031	040.0	0841.	0015.0
I 7	2670.	130.	0.0031	120.0	0841.	0045.0
I 8	2670.	130.	0.0031	240.0	0841.	0090.0
I 9*	2670.	130.	0.0031	400.0	0841.	0150.0
I10	2670.	130.	0.0031	1200.0(1)	0841.	0450.0
I11	2670.	130.	0.0031	2400.0(1)	0841.	0900.0
I12	2670.	130.	0.0031	4000.0(1)	0841.	1500.0

Table 9. Tabulation of damping constants for type I structure

Case No.	f structure	C_2/M_2	f translation	K_1	C_2/M_1	C_1/M_1	f rotation	K_r	C_r/I_T
1A	0.02	2.08	0.02	0.845×10^4	0.654	0.0872	0.02	30×10^6	0.0428
1B	0.02	2.08	0.05	0.845×10^4	0.654	0.218	0.05	30×10^6	0.107
1C	0.02	2.08	0.10	0.845×10^4	0.654	0.436	0.10	30×10^6	0.215
1D	0.02	2.08	0.20	0.845×10^4	0.654	0.872	0.20	30×10^6	0.430
1E	0.02	2.08	0.40	0.845×10^4	0.654	1.744	0.40	30×10^6	0.860
2A	0.02	2.08	0.02	1.69×10^4	0.654	0.1370	0.02	60×10^6	0.0660
2B	0.02	2.08	0.05	1.69×10^4	0.654	0.343	0.05	60×10^6	0.165
2C	0.02	2.08	0.10	1.69×10^4	0.654	0.686	0.10	60×10^6	0.310
2D	0.02	2.08	0.20	1.69×10^4	0.654	1.372	0.20	60×10^6	0.620
2E	0.02	2.08	0.40	1.69×10^4	0.654	2.744	0.40	60×10^6	1.240
3A	0.02	2.08	0.02	2.82×10^4	0.654	0.1800	0.02	100×10^6	0.0800
3B	0.02	2.08	0.05	2.82×10^4	0.654	0.4520	0.05	100×10^6	0.2000
3C	0.02	2.08	0.10	2.82×10^4	0.654	0.9040	0.10	100×10^6	0.4000
3D	0.02	2.08	0.20	2.82×10^4	0.654	1.8080	0.20	100×10^6	0.8000
3E	0.02	2.08	0.40	2.82×10^4	0.654	3.6160	0.40	100×10^6	1.6000
4A	0.02	2.08	0.02	8.45×10^4	0.654	0.2750	0.20	300.0×10^6	0.1345
4B	0.02	2.08	0.05	8.45×10^4	0.654	0.689	0.05	300.0×10^6	0.328
4C	0.02	2.08	0.10	8.45×10^4	0.654	1.378	0.10	300.0×10^6	0.656
4D	0.02	2.08	0.20	8.45×10^4	0.654	2.756	0.20	300.0×10^6	1.312
4E	0.02	2.08	0.40	8.45×10^4	0.654	5.512	0.40	300.0×10^6	2.624
5A	0.02	2.08	0.02	1.69×10^5	0.654	0.4320	0.02	600×10^6	0.0207
5B	0.02	2.08	0.05	1.69×10^5	0.654	1.080	0.05	600×10^6	0.518
5C	0.02	2.08	0.10	1.69×10^5	0.654	2.170	0.10	600×10^6	0.136
5D	0.02	2.08	0.20	1.69×10^5	0.654	4.330	0.20	600×10^6	0.272
5E	0.02	2.08	0.40	1.69×10^5	0.654	8.686	0.40	600×10^6	0.544

Table 9. (Continued)

Case No.	f structure	C_2/M_2	f translation	K_1	C_2/M_1	C_1/M_1	f rotation	K_r	C_r/I_T
6A	0.02	2.08	0.02	2.82×10^5	0.654	0.564	0.02	1×10^9	0.253
6B	0.02	2.08	0.05	2.82×10^5	0.654	1.410	0.05	1×10^9	0.631
6C	0.02	2.08	0.10	2.82×10^5	0.654	2.820	0.10	1×10^9	1.262
6D	0.02	2.08	0.20	2.82×10^5	0.654	5.640	0.20	1×10^9	2.524
6E	0.02	2.08	0.40	2.82×10^5	0.654	11.280	0.40	1×10^9	5.048
7A	0.02	2.08	0.02	8.45×10^5	0.654	0.872	0.02	3×10^9	0.428
7B	0.02	2.08	0.05	8.45×10^5	0.654	2.180	0.05	3×10^9	1.070
7C	0.02	2.08	0.10	8.45×10^5	0.654	4.360	0.10	3×10^9	2.150
7D	0.02	2.08	0.20	8.45×10^5	0.654	8.720	0.20	3×10^9	4.300
7E	0.02	2.08	0.40	8.45×10^5	0.654	17.440	0.40	3×10^9	8.600
8A	0.02	2.08	0.02	1.69×10^6	0.654	1.370	0.02	6×10^9	0.660
8B	0.02	2.08	0.05	1.69×10^6	0.654	3.430	0.05	6×10^9	1.650
8C	0.02	2.08	0.10	1.69×10^6	0.654	6.860	0.10	6×10^9	3.100
8D	0.02	2.08	0.20	1.69×10^6	0.654	13.720	0.20	6×10^9	6.200
8E	0.02	2.08	0.40	1.69×10^6	0.654	27.440	0.40	6×10^9	12.400
9A	0.02	2.08	0.02	2.82×10^6	0.654	1.810	0.02	10×10^9	0.800
9B	0.02	2.08	0.05	2.82×10^6	0.654	4.520	0.05	10×10^9	2.000
9C	0.02	2.08	0.10	2.82×10^6	0.654	9.040	0.10	10×10^9	4.000
9D	0.02	2.08	0.20	2.82×10^6	0.654	18.080	0.20	10×10^9	8.000
9E	0.02	2.08	0.40	2.82×10^6	0.654	36.160	0.40	10×10^9	16.000
10A	0.02	2.08	0.02	8.45×10^6	0.654	2.750	0.02	30.0×10^9	1.345
10B	0.02	2.08	0.05	8.45×10^6	0.654	6.890	0.05	30.0×10^9	3.280
10C	0.02	2.08	0.10	8.45×10^6	0.654	13.780	0.10	30.0×10^9	6.560
10D	0.02	2.08	0.20	8.45×10^6	0.654	27.560	0.20	30.0×10^9	13.120
10E	0.02	2.08	0.40	8.45×10^6	0.654	55.120	0.40	30.0×10^9	26.240
11A	0.02	2.08	0.02	1.69×10^7	0.654	4.320	0.02	60.0×10^9	2.070
11B	0.02	2.08	0.05	1.69×10^7	0.654	10.800	0.05	60.0×10^9	5.180
11C	0.02	2.08	0.10	1.69×10^7	0.654	21.700	0.10	60.0×10^9	13.600

Table 9. (Continued)

Case No.	f structure	C_2/M_2	f translation	K_1	C_2/M_1	C_1/M_1	f rotation	K_r	C_r/I_T
11D	0.02	2.08	0.20	1.69×10^{7}	0.654	43.300	0.20	60.0×10^{9}	27.200
11E	0.02	2.08	0.40	1.69×10^{7}	0.654	86.860	0.40	60.0×10^{9}	54.000
12A	0.02	2.08	0.02	2.82×10^{7}	0.654	5.640	0.02	100.0×10^{9}	2.530
12B	0.02	2.08	0.05	2.82×10^{7}	0.654	14.100	0.05	100.0×10^{9}	6.310
12C	0.02	2.08	0.10	2.82×10^{7}	0.654	28.200	0.10	100.0×10^{9}	12.620
12D	0.02	2.08	0.20	2.82×10^{7}	0.654	56.400	0.20	100.0×10^{9}	25.240
12E	0.02	2.08	0.40	2.82×10^{7}	0.654	112.800	0.40	100.0×10^{9}	50.480

Table 10. Tabulation of selected ratios, type II structure*

No base translation			No base rotation		
K_r	K_r/K_2	$K_r/M_2H_2^2+I_0$	K_1	K_1/K_2	K_1/M_1
30.0×10^6	53.40	000.477	0.845×10^4	.015	0003.45
60.0×10^6	106.80	000.954	1.69×10^4	.030	0006.89
100.0×10^6	178.0	001.59	2.82×10^4	.051	0011.51
300.0×10^6	534.0	004.77	8.45×10^4	.150	0034.5
600.0×10^6	1068.0	009.54	1.69×10^5	.30	0068.9
1.0×10^9	1780.0	015.9	2.82×10^5	.510	0115.1
3.0×10^9	5340.0	047.7	8.45×10^5	1.5	0345.0
6.0×10^9	10680.0	095.4	1.69×10^6	3.0	0689.0
10.0×10^9	17800.0	159.0	2.82×10^6	5.1	1151.0
30.0×10^9	53400.0	477.0	8.45×10^6	15.0	3450.0
60.0×10^9	106800.0	954.0	1.69×10^7	30.0	6890.0
100.0×10^9	178000.0	1590.0	2.82×10^7	150.0	11510.0

*Note $K_2 = 0.56 \times 10^6$

$$M_2H_2^2+I_0 = 62.6 \times10^6$$

$$M_1 = 2450$$

$$\frac{1}{M_1} = 4.08 \times 10^{-4}$$

$$\frac{1}{K_2} = 1.78 \times 10^{-6}$$

$$\frac{1}{62.6\times10^6} = .0159 \times 10^{-6}$$

Translation/rotation				
K_1	K_1/K_2	K_r	K_1/M_1	$K_r/M_2H_2^2+I_0$
0.845×10^4	.01500	30.0×10^6	0003.45	000.477
1.69×10^4	.03000	60.0×10^6	0006.89	000.954
2.82×10^4	.0501	100.0×10^6	0011.51	001.59
0.845×10^5	.1500	300.0×10^6	0034.5	004.77
1.69×10^5	.300	600.0×10^6	0068.9	009.54
2.82×10^5	.501	1.0×10^9	0115.1	015.9
8.45×10^5	1.5000	3.0×10^9	0345.0	047.7
1.69×10^6	3.0000	6.0×10^9	0689.0	095.7
2.82×10^6	5.010	10.0×10^9	1151.0	159.0
8.45×10^6	15.000	30.0×10^9	3450.0	477.0
1.69×10^7	30.000	60.0×10^9	6890.0	954.0
2.82×10^7	5.0100	100.0×10^9	11510.0	1590.0

Table 11. Tabulation of constants for type II structure, two free mass analysis, no damping

Case Number	K_2/M_2	H_2	M_2H_2/I_T	K_r/I_T	K_2/M_1	K_1/M_1
II 1	0734	250	0.0030	0477.	0.0228	00003.45
II 2	0734	250	0.0030	0954.	0.0228	00006.89
II 3*	0734	250	0.0030	0001.6	0.0228	01151.
II 4	0734	250	0.0030	0004.77	0.0228	00345.
II 5	0734	250	0.0030	0009.54	0.0228	00689.
II 6*	0734	250	0.0030	0016.0	0.0228	01151.
II 7	0734	250	0.0030	0047.7	0.0228	03450.
II 8	0734	250	0.0030	0095.7	0.0228	06890.
II 9*	0734	250	0.0030	0160.0	0.0228	01151.
II10	0734	250	0.0030	0477.0	0.0228	03450.
II11	0734	250	0.0030	0954.0	0.0228	06890.
II12	0734	250	0.0030	1590.0	0.0228	11510.

Table 12. Tabulation of damping constants for type II structure

Case No.	f structure	C_2/M_2	f translation	K_1	C_2/M_1	C_1/M_1	f rotation	K_r	C_r/I_T
1A	0.02	01.09	0.02	0.845×10^{4}	0.339	0.0851	0.02	30.0×10^{6}	0.02776
1B	0.02	01.09	0.05	0.845×10^{4}	0.339	0.213	0.05	30.0×10^{6}	0.069
1C	0.02	01.09	0.10	0.845×10^{4}	0.339	0.427	0.10	30.0×10^{6}	0.138
1D	0.02	01.09	0.20	0.845×10^{4}	0.339	0.854	0.20	30.0×10^{6}	0.276
1E	0.02	01.09	0.40	0.845×10^{4}	0.339	1.704	0.40	30.0×10^{6}	0.552
2A	0.02	01.09	0.02	1.69×10^{4}	0.339	0.1210	0.02	60.0×10^{6}	0.0392
2B	0.02	01.09	0.05	1.69×10^{4}	0.339	0.301	0.05	60.0×10^{6}	0.098
2C	0.02	01.09	0.10	1.69×10^{4}	0.339	0.602	0.10	60.0×10^{6}	0.196
2D	0.02	01.09	0.20	1.69×10^{4}	0.339	1.204	0.20	60.0×10^{6}	0.392
2E	0.02	01.09	0.40	1.69×10^{4}	0.339	2.408	0.40	60.0×10^{6}	0.784
3A	0.02	01.09	0.02	2.82×10^{4}	0.339	0.155	0.02	100.0×10^{6}	0.0509
3B	0.02	01.09	0.05	2.82×10^{4}	0.339	0.387	0.05	100.0×10^{6}	0.1272
3C	0.02	01.09	0.10	2.82×10^{4}	0.339	0.775	0.10	100.0×10^{6}	0.2545
3D	0.02	01.09	0.20	2.82×10^{4}	0.339	1.450	0.20	100.0×10^{6}	0.5090
3E	0.02	01.09	0.40	2.82×10^{4}	0.339	2.900	0.40	100.0×10^{6}	1.0180
4A	0.02	01.09	0.02	8.45×10^{4}	0.339	0.2680	0.02	300×10^{6}	0.0872
4B	0.02	01.09	0.05	8.45×10^{4}	0.339	0.6700	0.05	300×10^{6}	0.218
4C	0.02	01.09	0.10	8.45×10^{4}	0.339	1.3400	0.10	300×10^{6}	0.436
4D	0.02	01.09	0.20	8.45×10^{4}	0.339	2.69	0.20	300×10^{6}	0.872
4E	0.02	01.09	0.40	8.45×10^{4}	0.339	5.4000	0.40	300×10^{6}	1.744
5A	0.02	01.09	0.02	1.69×10^{5}	0.339	0.3810	0.02	600×10^{6}	0.124
5B	0.02	01.09	0.05	1.69×10^{5}	0.339	0.9470	0.05	600×10^{6}	0.310
5C	0.02	01.09	0.10	1.69×10^{5}	0.339	1.8940	0.10	600×10^{6}	0.620
5D	0.02	01.09	0.20	1.69×10^{5}	0.339	3.7880	0.20	600×10^{6}	1.040
5E	0.02	01.09	0.40	1.69×10^{5}	0.339	7.5760	0.40	600×10^{6}	2.080
6A	0.02	01.09	0.02	2.82×10^{5}	0.339	0.490	0.02	1000×10^{6}	0.1610
6B	0.02	01.09	0.05	2.82×10^{5}	0.339	1.220	0.05	1000×10^{6}	0.4025
6C	0.02	01.09	0.10	2.82×10^{5}	0.339	2.440	0.10	1000×10^{6}	0.8050
6D	0.02	01.09	0.20	2.82×10^{5}	0.339	4.880	0.20	1000×10^{6}	1.6100
6E	0.02	01.09	0.40	2.82×10^{5}	0.339	9.760	0.40	1000×10^{6}	3.2200

Table 12. (Continued)

Case No.	f structure	C_2/M_2	f translation	K_1	C_2/M_1	C_1/M_1	f rotation	K_r	C_r/I_T
7A	0.02	01.09	0.02	8.45×10^5	0.339	0.851	0.02	3.0×10^9	0.276
7B	0.02	01.09	0.05	8.45×10^5	0.339	2.130	0.05	3.0×10^9	0.690
7C	0.02	01.09	0.10	8.45×10^5	0.339	4.270	0.10	3.0×10^9	1.380
7D	0.02	01.09	0.20	8.45×10^5	0.339	8.540	0.20	3.0×10^9	2.760
7E	0.02	01.09	0.40	8.45×10^5	0.339	17.04	0.40	3.0×10^9	5.520
8A	0.02	01.09	0.02	1.69×10^6	0.339	1.210	0.02	6.0×10^9	0.392
8B	0.02	01.09	0.05	1.69×10^6	0.339	3.010	0.05	6.0×10^9	0.980
8C	0.02	01.09	0.10	1.69×10^6	0.339	6.02	0.10	6.0×10^9	1.960
8D	0.02	01.09	0.20	1.69×10^6	0.339	12.04	0.20	6.0×10^9	3.920
8E	0.02	01.09	0.40	1.69×10^6	0.339	24.08	0.40	6.0×10^9	7.840
9A	0.02	01.09	0.02	2.82×10^6	0.339	1.55	0.02	10.0×10^9	0.509
9B	0.02	01.09	0.05	2.82×10^6	0.339	3.87	0.05	10.0×10^9	1.272
9C	0.02	01.09	0.10	2.82×10^6	0.339	7.75	0.10	10.0×10^9	2.545
9D	0.02	01.09	0.20	2.82×10^6	0.339	14.50	0.20	10.0×10^9	5.090
9E	0.02	01.09	0.40	2.82×10^6	0.339	29.00	0.40	10.0×10^9	10.180
10A	0.02	01.09	0.02	8.45×10^6	0.339	2.680	0.02	30.0×10^9	0.872
10B	0.02	01.09	0.05	8.45×10^6	0.339	6.700	0.05	30.0×10^9	2.180
10C	0.02	01.09	0.10	8.45×10^6	0.339	13.400	0.10	30.0×10^9	4.360
10D	0.02	01.09	0.20	8.45×10^6	0.339	26.900	0.20	30.0×10^9	8.720
10E	0.02	01.09	0.40	8.45×10^6	0.339	54.000	0.40	30.0×10^9	17.440
11A	0.02	01.09	0.02	1.69×10^7	0.339	3.810	0.02	60.0×10^9	1.24
11B	0.02	01.09	0.05	1.69×10^7	0.339	9.470	0.05	60.0×10^9	3.10
11C	0.02	01.09	0.10	1.69×10^7	0.339	18.940	0.10	60.0×10^9	6.20
11D	0.02	01.09	0.20	1.69×10^7	0.339	37.880	0.20	60.0×10^9	10.40
11E	0.02	01.09	0.40	1.69×10^7	0.339	75.760	0.40	60.0×10^9	20.80
12A	0.02	01.09	0.02	2.82×10^7	0.339	4.900	0.02	100.0×10^9	1.610
12B	0.02	01.09	0.05	2.82×10^7	0.339	12.200	0.05	100.0×10^9	4.025
12C	0.02	01.09	0.10	2.82×10^7	0.339	24.400	0.10	100.0×10^9	8.050
12D	0.02	01.09	0.20	2.82×10^7	0.339	48.800	0.20	100.0×10^9	16.100
12E	0.02	01.09	0.40	2.82×10^7	0.339	97.600	0.40	100.0×10^9	32.200